Proof of perturbative gauge invariance for tree diagrams to all orders

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Abstract

It is proved that classical BRS-invariance of the Lagrangian implies perturbative gauge invariance for tree diagrams to all orders. The proof applies in particular to the Einstein Hilbert Largrangian of gravity.

PACS. 11.15.Bt Gauge field theories: General properties of perturbation theory

Contents

1 Introduction

One of the greatest challenges of present-day quantum field theory (QFT) is the search for a quantum theory of gravity. In this context revolutionary approaches are intensively studied, e.g. non-commutative space-times, string theory and loop quantum gravity. Since this paper is related to BRS-symmetry [3], we only mention that a BRS-formulation of gravity was given in [5, 6, 31] and that the structure of the possible anomalies has been worked out by cohomological methods, see e.g. [4, 2].

The main result of this paper is much more modest: we prove *perturbative gauge invariance* (PGI) [14, 26, 27, 17, 30] (which is a condition in perturbative QFT that is related to BRS-invariance) for gravity, but our result is restricted to *tree diagrams*. Since PGI for tree diagrams (PGI-tree) is equivalent to PGI in *classical* field theory (cf. Appendix B of [9] and Sect. 2), our result is actually a statement for classical gravity. However, it is also a justification to use PGI for the construction of a perturbative QFT for spin-2 gauge fields - a project started in [29, 28, 20, 27, 18, 19].

In the latter the requirement of PGI-tree has been used to determine the possible interactions of massless spin-2 gauge fields [29, 28, 27]. Making a polynomial ansatz for the interaction $\mathcal{L} = \sum_{n=1}^{\infty} \kappa^n \mathcal{L}^{(n)}$ (where κ is the coupling constant), it has been worked out that the most general solutions for $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ agree with the corresponding terms of the Einstein-Hilbert Lagrangian of gravity up to physically irrelevant terms, see [28, 27]. But continuing this procedure to higher orders the amount of computational work increases strongly and, due to the non-renormalizability of spin-2 gauge fields, one never comes to an end. That is, violations of PGI-tree can appear to arbitrary high orders and, if PGI-tree can be fulfilled, it is not clear, that the general solution for $\mathcal{L}^{(k)}$, $k \geq 3$, agrees with the corresponding term of the Einstein-Hilbert Lagrangian. The *main purpose of this paper is to prove that the Einstein-Hilbert Lagrangian, completed by a gauge fixing and a Faddeev-Popov ghost term* (we follow [23]), *yields a solution of PGI-tree to*

all orders.

To a large extent we formulate the proof independently of the model. However, the applications to massless and massive spin-1 fields (Sects. 4.1 and 4.2) are only of pedagogical value: for *renormalizable models* a violation of PGI-tree can usually[1] appear only up to third order (as can easily be seen by power counting), and using this fact, PGI-tree has been proved by explicit computation of the lowest orders for various spin-1 models [14, 15, 17].

By *perturbative gauge invariance* we mean the following condition. Let a free quantum gauge theory (i.e. a free Lagrangian $\mathcal{L}^{(0)}$) and the corresponding free BRS-transformation s_0 be given and let $\mathcal{L}^{(0)}$ be s_0 invariant, that is $s_0\,\mathcal{L}^{(0)} = -\partial_\mu I^{(0)\,\mu}$ for some field polynomial $I^{(0)\,\mu}$. In addition let $j^{(0)\,\mu}$ be the corresponding conserved Noether current, $\partial_\mu j^{(0)\,\mu}_{S_0} = 0$, and let Q be the corresponding charge: $Q = \int d^3x\, j^{(0)\,0}_{S_0}(x^0, \vec{x})$ ("free BRS-charge"). (The lower index S_0 signifies always that we mean the 'on-shell fields', i.e. the free field equations are valid; for a precise formulation see [9, 10] or Appendix A.) PGI requires that to an interaction $\mathcal{L}^{(1)}$ there exists a Lorentz vector $\mathcal{L}^{(1)\,\nu}_1$ and a normalization of the time ordered products T^N such that

$$[Q, T^N_{S_0}\big(\mathcal{L}^{(1)}(x_1)...\mathcal{L}^{(1)}(x_n)\big)] = i\sum_{l=1}^{n} \partial^{x_l}_\nu\, T^N_{S_0}\big(\mathcal{L}^{(1)}(x_1)...\mathcal{L}^{(1)\,\nu}_1(x_l)...\mathcal{L}^{(1)}(x_n)\big) \tag{1.1}$$

(where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator with respect to the $\star$ product). The upper index (1) of $\mathcal{L}^{(1)}$ signifies that we mean the term of first order (in the coupling constant κ) of the total interaction $\mathcal{L} = \sum_{n=1}^\infty \kappa^n\,\mathcal{L}^{(n)}$ and similarly $\mathcal{L}^{(1)\,\nu}_1$ is the term of first order of the total "Q-vertex" $\mathcal{L}^\nu_1 = \sum_{n=1}^\infty \kappa^n\,\mathcal{L}^{(n)\,\nu}_1$. Higher order terms $\mathcal{L}^{(n)}$ ($\mathcal{L}^{(n)\,\nu}_1$ resp.), $n \geq 2$, are absorbed in a finite renormalization $T(\mathcal{L}^{(1)}...\mathcal{L}^{(1)}) \to T^N(\mathcal{L}^{(1)}...\mathcal{L}^{(1)})$ ($T(\mathcal{L}^{(1)}...\mathcal{L}^{(1)\,\nu}_1...) \to T^N(\mathcal{L}^{(1)}...\mathcal{L}^{(1)\,\nu}_1...)$ resp.) of tree diagrams. This is always possible, as shown in Sect. 2.

PGI plays two different roles:

(i) PGI-tree restricts the interaction $\mathcal{L} = \sum_{n=1}^\infty \kappa^n\,\mathcal{L}^{(n)}$ strongly, as mentioned above.

(ii) For loop diagrams it is a highly non-trivial (re)normalization condition, which cannot always be fulfilled, e.g. in case of the axial anomaly.

[1] The statement is valid for renormalizable models in 4 dimensions with the property that all terms of the interaction $\mathcal{L}$ are at least of third order in the basic fields.

The motivations to require PGI are the following.

(A) In purely massive theories the adiabatic limit exists [16], i.e. there is an S-matrix

$$S = \mathbf{1} + \lim_{g\to 1}\sum_{n=1}^{\infty}\frac{i^n}{n!}\int dx_1...dx_n\, g(x_1)...g(x_n)\, T^N_{S_0}\big(\mathcal{L}^{(1)}(x_1)...\mathcal{L}^{(1)}(x_n)\big)\ . \tag{1.2}$$

Kugo and Ojima [22, 23] have shown that in the adiabatic limit the physical Hilbert space $\mathcal{H}$ can be expressed in terms of the free BRS-charge Q:

$$\mathcal{H} = \frac{\ker Q}{\operatorname{ran} Q}\ . \tag{1.3}$$

PGI implies $[Q,S]=0$ and, hence, S is well defined on $\mathcal{H}$. That is, PGI is a sufficient condition for the quantization of purely massive gauge theories and, as shown in [13], it is even almost necessary for this purpose.

(B) Since PGI is well defined also for theories in which the adiabatic limit does not exist, PGI-tree can be used to derive the interaction $\mathcal{L}$ for all kinds of gauge theories. Making a polynomial ansatz for $\mathcal{L}$, PGI-tree and some obvious requirements (e.g. Lorentz invariance, ghost number zero and in case of spin-1 fields renormalizability) determine $\mathcal{L}$ to a far extent. We recall the highlights (besides the already mentioned derivation of the Einstein-Hilbert Lagrangian).

- The Lie algebraic structure of spin-1 fields needs not to be put in, it can be derived in this way [30].
- For non-Abelian massive spin-1 theories it is impossible to satisfy these requirements for a model with only gauge fields and ghosts (fermionic and bosonic). The inclusion of additional *physical* scalar fields (corresponding to Higgs fields) yields a solution [15].

In this paper we proceed in the direction opposite to (B): we assume that a Lagrangian $\mathcal{L}_{\text{total}} = \sum_{n=0}^{\infty}\kappa^n\,\mathcal{L}^{(n)}$ and a BRS-transformation $s = \sum_{n=0}^{\infty}\kappa^n\,s_n$ (of the interacting fields) [3] are given and that $\mathcal{L}_{\text{total}}$ is BRS-invariant: $s\,\mathcal{L}_{\text{total}} = -\partial_\mu I^\mu$ for some formal power series I^μ. We prove that this assumption implies PGI-tree.

Our proof of PGI-tree relies strongly on results which have been found in [7], [12] and [9]. Namely, let us start with the local conservation of the BRS-current,

$$\partial^x_\mu T^N_{S_0}\big(j^{(0)\,\mu}(x),\mathcal{L}^{(1)}(x_1)...\mathcal{L}^{(1)}(x_n)\big) =$$
$$-i\sum_{l=1}^n \partial^{x_l}_\nu \Big(\delta(x-x_l)\,T^N_{S_0}\big(\mathcal{L}^{(1)}(x_1)...\mathcal{L}^{(1)\,\nu}_1(x_l)...\mathcal{L}^{(1)}(x_n)\big)\Big) \tag{1.4}$$

where the higher order terms $j^{(n)}$, $n\geq 1$, of the total BRS-current $j^\mu = \sum_{n=0}^\infty \kappa^n\, j^{(n)\,\mu}$ are absorbed in a renormalization $T(j^{(0)},\mathcal{L}^{(1)}...)\to T^N(j^{(0)},\mathcal{L}^{(1)}...)$ of tree diagrams (see Sect. 2). In Appendix B of [7] and in [12] it is proved that by smearing out (1.4) with a test function $f(x)$ which satisfies $f|_{\bar{\mathcal{O}}}=1$, where $\mathcal{O}$ is an open double cone containing $x_1,...,x_n$, one obtains PGI (1.1). It has even been shown that (1.4) is necessary for PGI provided the ghost number is conserved (Sect. 4.5.2 of [12], related ideas are given in [21]). Motivated by these facts we proceed as follows. In (non-perturbative) classical field theory we show that BRS-invariance of the Lagrangian (for constant coupling) implies local conservation of the BRS-current. The latter holds also for the perturbative expansion of the classical fields, i.e. for the retarded product of classical field theory $R^{\rm class}$. Since $R^{\rm class}$ agrees with the contribution of the tree diagrams $R^{\rm tree}$ to the retarded product of QFT [8, 9], we obtain the translation of (1.4) into $R^{N\,\rm tree}$, after a finite renormalization $R\to R^N$ (of tree diagrams). Then, proceeding analogously to the step from (1.4) to PGI (see Sect. 5.2 of [9]), we obtain PGI-tree for $R^{N\,\rm tree}$. (Up to third order this result is derived also in Appendix B of [9].) Finally we show that PGI-tree is maintained in the transition to the corresponding time ordered product T^N (by using results about the counting of powers of $\hbar$ given in Sect. 5 of [8]).

The paper is organized as follows. In Sect. 2 we assume that the *classical* BRS-current is locally conserved (2.24), and give a model independent proof that this implies PGI-tree. In Sect. 3 we trace back this assumption to BRS-invariance of the Lagrangian for constant coupling, still independently of the model. In doing so we obtain explicit formulas for the BRS-current j^μ and the Q-vertex $\mathcal{L}^\nu_1$.

We then illustrate the formalism: for massless and massive Yang-Mills theories we find that our formulas for j^μ and $\mathcal{L}^\nu_1$ yield results which agree with the literature (Sect. 4.1 and 4.2).

Then, we turn to the main objective of this paper: massless spin-2 gauge fields (Sect. 4.3). From Kugo and Ojima [23] we recall the BRS-invariance of the Lagrangian of gravity and show that it fits in our formalism. This completes our proof of PGI-tree to all orders for massless spin-2 gauge fields. We verify that our formula for $j^{(0)\,\mu}$ agrees with the literature also in the spin-2 case.

2 From classical current conservation to perturbative gauge invariance for tree diagrams

Let $\mathcal{P}$ be the polynomial algebra generated by the basic classical (off-shell) fields and their partial derivatives, see Appendix A. In this Sect. we assume that there are given

- an action $S_{\rm total}(g) = S_0 + S(g)$ with free part $S_0 = \hbar^{-1}\int dx\,\mathcal{L}^{(0)}(x)$ and interacting part

$$S(g) = \int dx\,\mathcal{L}(g)(x)\ ,\quad \mathcal{L}(g)(x) := \hbar^{-1}\sum_{k=1}^{\infty}\kappa^k\,(g(x))^k\,\mathcal{L}^{(k)}(x)\ ,\tag{2.1}$$

  $\mathcal{L}^{(k)} \in \mathcal{P}$ ($\forall k = 0,1,2,3,...$), where κ is the coupling constant and $g \in \mathcal{D}(\mathbb{R}^4)$ is a test function which switches the interaction;

- a BRS-current

$$j^\mu(g)(x) := \sum_{k=0}^{\infty}\kappa^k\,(g(x))^k\,j^{(k)\,\mu}(x)\ ,\quad j^{(k)\,\mu}\in\mathcal{P}\ ;\tag{2.2}$$

- and a Q-vertex[2]

$$\mathcal{L}_1^\nu(g)(x) := \sum_{k=1}^{\infty}\kappa^k\,(g(x))^{(k-1)}\,\mathcal{L}_1^{(k)\,\nu}(x)\ ,\quad \mathcal{L}_1^{(k)\,\nu}\in\mathcal{P}\ .\tag{2.3}$$

[2]In the terminology of [27] the defining property of a Q-vertex $\mathcal{L}_1^{(1)\,\nu}$ is (1.1) for $n=1$. The fact that we use the word "Q-vertex" does not mean that we assume this identity to hold, it will be part of our conclusion. In addition, in our terminology a Q-vertex contains also terms of higher orders in κ.

Usually $\mathcal{L}^{(k)}$, $j^{(k)\,\mu}$ and $\mathcal{L}_1^{(k)\,\nu}$ are of $(k+2)$-th order in the basic fields. The really restricting part of our assumption is that in *classical field theory* $j^\mu(g)$ and $\mathcal{L}_1^\nu(g)$ are related by a certain local current conservation (see (2.24) below) which is a consequence of the field equations given by $S_{\rm total}(g)$. We show that this implies PGI-tree, i.e. the equation (1.1) with T^N replaced by the contribution $T^{N\,{\rm tree}}$ of its tree diagrams (on both sides of (1.1)).

We use the formalism of [10] and [9], see Appendix A. With that a perturbative classical field (A.7) agrees exactly with the contribution of the tree diagrams $A^{\rm tree}_{S(g)}(x)$ to the corresponding field $A_{S(g)}(x)$ (A.9) of perturbative QFT: due to $A\sim\hbar^0$, $S(g)\sim\hbar^{-1}$ and (A.13) it holds [8]

$$A_{S(g)}(x)=A^{\rm tree}_{S(g)}(x)+\mathcal{O}(\hbar)\quad\text{and}\quad A^{\rm tree}_{S(g)}(x)\sim\hbar^0\ , \tag{2.4}$$

and hence

$$A^{\rm tree}_{S(g)}(x)|_{\mathcal{C}_{S_0}}\equiv R^{\rm tree}_{S_0}\Big(e_\otimes^{S(g)},A(x)\Big)=R^{\rm class}_{S_0}\Big(e_\otimes^{S(g)},A(x)\Big)\ . \tag{2.5}$$

Due to this identity the classical factorization of composite fields holds also for $A^{\rm tree}_{S(g)}(x)$ [9],

$$(AB)^{\rm tree}_{S(g)}(x)=A^{\rm tree}_{S(g)}(x)\cdot B^{\rm tree}_{S(g)}(x)\ ,\quad A,B\in\mathcal{P}\ , \tag{2.6}$$

and the fields $\varphi^{\rm tree}_{S(g)}(x)|_{\mathcal{C}_{S_0}}$ (where φ runs through the basic fields) satisfy the classical field equations, which form a *closed* system of partial differential equations. The product on the right side of (2.6) is the classical product (see Appendix A), not the $\star$-product (A.8). The latter is related to the Poisson bracket (of classical field theory) by

$$\{F,G\}=\lim_{\hbar\to 0}\frac{i}{\hbar}(F\star G-G\star F)\ ,\quad F,G\in\mathcal{F}\ . \tag{2.7}$$

Next we give some preparations concerning the BRS-structure. In a BRS-model [3] the basic fields are

- bosonic gauge fields with spin-1 (A^μ) or spin-2 ($h^{\mu\nu}$),
- for each gauge field there is precisely one pair of fermionic ghost fields $(u,\tilde{u})$,
- for each *massive* gauge field there is precisely one bosonic ghost field ϕ,

- in non-Abelian massive spin-1 gauge theories there is at least one physical Higgs field H
- and there may be fermionic spinor fields.

The field algebras $\mathcal{F}$, $\mathcal{F}_0^{(m)} \equiv \frac{\mathcal{F}}{\mathcal{J}^{(m)}}$ (classical product, see Appendix A) and $\mathcal{A}_0^{(m)} \equiv (\mathcal{F}_0^{(m)}, \star_m)$ are $\mathbb{Z}_2$-graded by the number of ghost fields: $\mathcal{F} = \mathcal{F}_{\text{even}} \oplus \mathcal{F}_{\text{odd}}$. The ghost number of the action is even, and it is odd for the BRS-current and the Q-vertex.

We assume that the free BRS-transformation s_0 acts linearly on the basic fields, i.e. $s_0\varphi$ (φ a basic field) is a linear combination of partial derivatives of basic fields. This implies that $j^{(0)\,\mu}$ is quadratic in the (derivated) basic fields, since $\mathcal{L}^{(0)}$ is quadratic in the (derivated) basic fields.

We shall need some basic properties of the free BRS-charge Q (or more precisely of d_Q (2.11)). Formally, Q is given by

$$Q := \int_{x^0=\text{const.}} d^3x \, j_{S_0}^{(0)\,0}(x^0, \vec{x}) \ , \tag{2.8}$$

which is a functional on $\mathcal{C}_{S_0}$. The problem with this formula is that a priori $j_{S_0}^{(0)\,0}(x^0, \vec{x})$ can only be integrated out in $\vec{x}$ *and* x^0 and only with a test function. To give a rigorous definition we follow Sect. 5.1 of [7] where a method of Requardt [25] is used. Let $k(x^0)\, h(\vec{x}) \in \mathcal{D}(\mathbb{R}^4)$, where $\int dx^0\, k(x^0) = 1$ and h is a smeared characteristic function of $\{\vec{x} \in \mathbb{R}^3, |\vec{x}| \leq R\}$ for some $R > 0$. We scale the test function such that the normalization of k is maintained,

$$k_\lambda(x^0) \overset{\text{def}}{=} \lambda\, k(\lambda x^0) \ , \quad h_\lambda(\vec{x}) \overset{\text{def}}{=} h(\lambda \vec{x}) \ , \tag{2.9}$$

and want to define Q as the limit

$$Q \overset{\text{def}}{=} \lim_{\lambda \to 0} Q_\lambda \ , \quad Q_\lambda \overset{\text{def}}{=} \int d^4x\, k_\lambda(x^0)\, h_\lambda(\vec{x})\, j_{S_0}^{(0)\,0}(x^0, \vec{x}) \ . \tag{2.10}$$

As far as we know the existence of this limit cannot be shown generally, structural information about the concrete model is needed. However, in this paper we are not interested in Q itself, we only study the operator $d_Q : \mathcal{A}_0^{(m)} \to \mathcal{A}_0^{(m)}$, which is given by the graded commutator[3]

$$d_Q\, F_{S_0} \overset{\text{def}}{=} \lim_{\lambda \to 0} (Q_\lambda \star F_{S_0} \mp F_{S_0} \star Q_\lambda) =: \lim_{\lambda \to 0} [Q_\lambda, F_{S_0}]_\star^\mp \ , \tag{2.11}$$

[3]Note that the ghost number of Q is odd.

where the minus sign appears for $F \in \mathcal{F}_{\rm even}$ and the plus sign for $F \in \mathcal{F}_{\rm odd}$. It immediately follows that d_Q is a graded derivation[4] with respect to the $\star$-product, provided the limit

$$d_Q\, F_{S_0} = \lim_{\lambda\to 0}\, [Q_\lambda, F_{S_0}]_\star^\mp = \lim_{\lambda\to 0} \int dx^0\, k_\lambda(x^0) \int d^3x\, h_\lambda(\vec{x})\, [j^{(0)\,0}_{S_0}(x^0,\vec{x}), F_{S_0}]_\star^\mp \tag{2.13}$$

exists. This holds indeed true [7]: namely, because of supp $[j^{(0)\,0}_{S_0}(x), F_{S_0}]_\star^\mp \subset (\mathrm{supp}\, F + (\bar{V}_+ \cup \bar{V}_-))$ we may replace $h_\lambda(\vec{x})$ by 1 for $\lambda > 0$ sufficiently small and R big compared with the support of k. Note that $\int d^3x\, [j^{(0)\,0}_{S_0}(x^0,\vec{x}), F_{S_0}]_\star^\mp$ exists since the region of integration is bounded; and, due to current conservation, it is independent of x^0. This yields

$$\lim_{\lambda\to 0}\, [Q_\lambda, F_{S_0}]_\star^\mp = \int_{x^0=\mathrm{const.}} d^3x\, [j^{(0)\,0}_{S_0}(x^0,\vec{x}), F_{S_0}]_\star^\mp \ . \tag{2.14}$$

This result holds for the terms $\sim \hbar$ of $[\,\cdot\,,\,\cdot\,]_\star^\mp$ separately. Hence, we may define

$$\{Q, F_{S_0}\} :\stackrel{\mathrm{def}}{=} \lim_{\lambda\to 0}\, \{Q_\lambda, F_{S_0}\} = \lim_{\hbar\to 0} \frac{i}{\hbar}\, d_Q\, F_{S_0} \tag{2.15}$$

and obtain $\{Q, F_{S_0}\} = \int d^3x\, \{j^{(0)\,0}_{S_0}(x^0,\vec{x}), F_{S_0}\}$.

For the concrete models studied in Sect. 4 it holds

$$d_Q^2 = 0 \tag{2.16}$$

and

$$\omega_0(d_Q\, F_{S_0}) = 0\ , \quad \forall F \in \mathcal{F}\ , \tag{2.17}$$

as it is verified e.g. in [27] (in a Krein-Fock space representation). In this paper we assume only the validity of (2.17), the nil-potency will not be needed. (Usually, the fields are represented on an inner product space such that $\langle F^*\Phi, \Psi\rangle = \langle \Phi, F\Psi\rangle$ (where $\langle\,\cdot\,,\,\cdot\,\rangle$ must be indefinite) and in that representation one proves that Q is a nilpotent and symmetric operator which annihilates the vacuum; these properties imply (2.16) and (2.17).)

We are now going to show that d_Q is a graded derivation also with respect to the **classical** product. This follows from the observation that in $d_Q\, F_{S_0}$ (2.11) solely the terms $\sim \hbar$ of the $\star$-product contribute. In detail:

[4] A graded derivation D of a $\mathbb{Z}_2$-graded algebra $\mathcal{A}$ is defined to be a **linear** map $D : \mathcal{A} \to \mathcal{A}$ with

$$D(A \cdot B) = D(A) \cdot B + (-1)^{\epsilon(A)}\, A \cdot D(B)\ , \tag{2.12}$$

where A is of definite degree $\epsilon(A) \in \{0, 1\}$.

Lemma 1. *(i)*

$$d_Q\, F_{S_0} = -i\hbar\, \{Q, F_{S_0}\}\ , \quad \forall F \in \mathcal{F}\ , \tag{2.18}$$

where $\{Q,\cdot\}$ is defined by (2.15).

(ii)

$$d_Q\, (F_{S_0}\, G_{S_0}) = d_Q(F_{S_0})\, G_{S_0} \pm F_{S_0}\, d_Q(G_{S_0})\ , \tag{2.19}$$

where the $+$-sign holds for $F \in \mathcal{F}_{\mathrm{even}}$ and the $-$-sign holds for $F \in \mathcal{F}_{\mathrm{odd}}$.

Proof. (ii) is a consequence of (i): since the graded Poisson bracket satisfies the graded Leibniz rule, the map $\{Q,\cdot\}$ (2.15) is a graded derivation with respect to the classical product.

To concentrate on the essential steps of the proof of (i) we replace F_{S_0} by $\phi_1 ... \phi_n$ (classical product), where ϕ_j is (a derivative of) a basic Bose field $\varphi_{i_j}(x_j)$: $\phi_j = \partial^{a_j}\varphi_{i_j}(x_j)_{S_0}$. All non-vanishing terms in $[Q_\lambda, \phi_j]_\star$ have one contraction, hence

$$d_Q\, \phi_j = -i\hbar\, \{Q, \phi_j\}\ . \tag{2.20}$$

To show that $d_Q(\phi_1 ... \phi_n)$ agrees with

$$-i\hbar\, \{Q, \phi_1 ... \phi_n\} = -i\hbar \sum_{k=1}^{n} \phi_1 ... \{Q, \phi_k\} ... \phi_n\ , \tag{2.21}$$

we proceed by induction on the number n of factors. By using

- the recursion relation

$$\phi_1 ... \phi_{n+1} = (\phi_1 ... \phi_n) \star \phi_{n+1} - \sum_{l=1}^{n} (\phi_1 ... \hat{l} ... \phi_n)\, \omega_0(\phi_l \star \phi_{n+1}) \tag{2.22}$$

  (which follows from the definition (A.8) of the $\star$-product),

- our assumption that s_0 acts linearly on the basic fields which implies that $\{Q, \phi_j\}$ is a linear combination of derivated basic fields,

- and

$$\omega_0(\{Q, \phi_l\} \star \phi_{n+1}) + \omega_0(\phi_l \star \{Q, \phi_{n+1}\}) = \frac{i}{\hbar}\, \omega_0([Q\ ,\ \phi_l \star \phi_{n+1}]_\star) = 0 \tag{2.23}$$

  (where (2.17) is used),

one verifies the inductive step straightforwardly. $\square$

An alternative proof of the statement (ii) can be found in Lemma 3.1.1. of [27].

In this Sect. we **assume** that there exist a BRS-current $j^\mu(g)$ (2.2), an action $S_{\text{total}}(g) = S_0 + S(g)$ (2.1) and a Q-vertex $\mathcal{L}_1^\nu(g)$ (2.3) such that they fulfill the local current conservation

$$\partial_\mu\, j^\mu(g)(x)_{S_{\text{total}}(g)} = (\partial_\nu g)(x)\, \mathcal{L}_1^\nu(g)(x)_{S_{\text{total}}(g)}\ , \tag{2.24}$$

in classical field theory. This identity implies that the *BRS-current* $j^\mu(g)(x)_{S_{\text{total}}(g)}$ *is conserved for* $x \in \mathcal{U}$ (where $\mathcal{U} \subset \mathbb{R}^4$ is an open set) *if* $g|_{\mathcal{U}}$ *is constant*. We will verify the assumption (2.24) for concrete models in the following Sects.. The current conservation (2.24) holds also for the corresponding retarded fields (A.6) and, hence, also for the perturbative expansion (A.7) of the latter. With that and due to the identity (2.5) we obtain a statement for the tree diagrams of perturbative QFT:

$$-R^{\text{tree}}_{S_0}\Big(e_\otimes^{S(g)}, \int j(g)^\mu\, \partial_\mu f\Big) = R^{\text{tree}}_{S_0}\Big(e_\otimes^{S(g)}, \int \mathcal{L}_1^\nu(g)\, f\, \partial_\nu g\Big)\ . \tag{2.25}$$

We are now going to absorb the higher order terms of $S(g)$, $j(g)^\mu$ and $\mathcal{L}_1^\nu(g)$ in an admissible renormalization of the retarded product by using the 'Main Theorem of Perturbative Renormalization' (Sect. 4.2 of [10]). We assume that $\mathcal{L}^{(1)}$, $j^{(0)\,\mu}$ and $\mathcal{L}_1^{(1)\,\nu}$ are linearly independent fields; or that $\mathcal{L}^{(1)}$ and $j^{(0)\,\mu}$ are linearly independent and $\mathcal{L}_1^{(1)\,\nu} = 0$. (This assumption seems to hold true in all cases of interest, see Sect. 4.) With that we set

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n!}\, D_n\Big(\big(\int g\,\mathcal{L}^{(1)}\big)^{\otimes n}\Big) &\equiv D\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}}\Big) \overset{\text{def}}{=} S(g)\ ,\\
D\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}} \otimes \int h_\mu\, j^{(0)\,\mu}\Big) &\overset{\text{def}}{=} \int dx\, h_\mu(x)\, j^\mu(g)(x)\ ,\\
D\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}} \otimes \int h_\nu\, \mathcal{L}_1^{(1)\,\nu}\Big) &\overset{\text{def}}{=} \int dx\, h_\nu(x)\, \mathcal{L}_1^\nu(g)(x)
\end{aligned} \tag{2.26}$$

and extend D_n to a *linear* and *symmetrical* map

$$D_n : \mathcal{F}_{\text{loc}}^{\otimes n} \longrightarrow \mathcal{F}_{\text{loc}} \tag{2.27}$$

with

- $D_0(1) = 0, \quad D_1(F) = F$,
- $$\text{supp}\, \frac{\delta\, D_n(F_1 \otimes \ldots \otimes F_n)}{\delta\varphi} \subset \bigcap_{i=1}^{n} \text{supp}\, \frac{\delta F_i}{\delta\varphi} \;, \tag{2.28}$$
- Poincaré covariance and
- $D(F^{\otimes n})^* = D((F^*)^{\otimes n})$.

Part (iv) of the Main Theorem then states that[5]

$$R^N(e_\otimes^{\lambda G}, F) \stackrel{\text{def}}{=} R(e_\otimes^{D(e_\otimes^{\lambda G})}, D(e_\otimes^{\lambda G} \otimes F)) \tag{2.29}$$

defines a new retarded product R^N, i.e. R^N fulfills the basic properties (A.10)-(A.12). And, if R is Poincaré covariant and unitary (i.e. $R(F^{\otimes n})^* = R((F^*)^{\otimes n})$), then these properties hold also for R^N. Usually $\omega_0(R_{n-1,1}(...))$ satisfies an upper bound on the scaling behaviour in the UV-region. The map D (2.27) can be chosen such that also this bound is maintained in the renormalization $R \longrightarrow R^N$. In terms of R^N the current conservation (2.25) reads

$$-R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}}, \int j^{(0)\,\mu}\,\partial_\mu f\Big) = R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}}, \int \mathcal{L}_1^{(1)\,\nu}\, f\,\partial_\nu g\Big) + \mathcal{O}(\hbar) \;, \tag{2.30}$$

where we have also used (A.13). This identity implies

$$\begin{aligned} -R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}} \otimes \int g\,\mathcal{L}^{(1)}, \int j^{(0)\,\mu}\,\partial_\mu f\Big) &= R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}}, \int \mathcal{L}_1^{(1)\,\nu}\, f\,\partial_\nu g\Big) \\ &+ R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}} \otimes \int g\,\mathcal{L}^{(1)}, \int \mathcal{L}_1^{(1)\,\nu}\, f\,\partial_\nu g\Big) + \mathcal{O}(\hbar) \;. \end{aligned} \tag{2.31}$$

We are now going to derive PGI for $R^{N\,\text{tree}}$. For a given $g \in \mathcal{D}(\mathbb{R}^4)$ let $\mathcal{O}$ be an open double cone such that $\text{supp}\, g \subset \mathcal{O}$. Furthermore we choose $f \in \mathcal{D}(\mathbb{R}^4)$ with $f \equiv 1$ on a neighborhood of $\overline{\mathcal{O}}$. We decompose $\partial_\mu f = b_\mu - a_\mu$ such that $\text{supp}\, b_\mu \cap (\mathcal{O} + \bar{V}_+) = \emptyset$ and $\text{supp}\, a_\mu \cap (\mathcal{O} + \bar{V}_-) = \emptyset$. In the following calculation (which is mainly taken from Sect. 5.2 of [9]) we take into account $\mathcal{L}^{(1)} \sim \hbar^{-1}$, $\mathcal{L}_1^{(1)} \sim \hbar^0$ and

$$\{Q, F_{S_0}\} = \{\int dx\, j_{S_0}^{(0)\,\mu}(x)\, b_\mu(x), F_{S_0}\} \;, \quad \forall F \in \mathcal{F}(\mathcal{O}) \;, \tag{2.32}$$

[5]This formula has to be understood in the sense of formal power series in λ.

which follows from the locality of the Poisson bracket (i.e. $\{G_{S_0},H_{S_0}\}=0$ if the supports of G_{S_0} and H_{S_0} are space-like separated), the current conservation $\partial_\mu j^{(0)\,\mu}_{S_0}=0$ and the definition of $\{Q,\cdot\}$ (2.15). (A detailed explanation of the same conclusion for the commutator is given in Appendix B of [7].) In addition we use Lemma 1(i), the characterization (A.13) of $R^{N\,\mathrm{tree}}$, $R_{0,1}(F)=F$, causality (A.11) and the GLZ relation (A.12):

$$\begin{aligned} d_Q\, R^{N\,\mathrm{tree}}_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}},\int g\,\mathcal{L}^{(1)}\Big) &= -i\hbar\Big\{\int j^{(0)\,\mu}_{S_0}\,b_\mu\,,\,R^{N\,\mathrm{tree}}_{S_0}\Big(...\Big)\Big\}\\ &=\Big[\int j^{(0)}_{S_0}\,b\,,\,R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}},\int g\,\mathcal{L}^{(1)}\Big)\Big]_\star|_{\hbar^0}\\ &=\Big[R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}},\int j^{(0)}\,b\Big)\,,\,R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}},\int g\,\mathcal{L}^{(1)}\Big)\Big]_\star|_{\hbar^0}\\ &= i\Big(R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}}\otimes\int j^{(0)}\,b,\int g\,\mathcal{L}^{(1)}\Big)-R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}}\otimes\int g\,\mathcal{L}^{(1)},\int j^{(0)}\,b\Big)\Big)|_{\hbar^0}\;, \end{aligned}\tag{2.33}$$

where $...|_{\hbar^0}$ signifies that we only mean the contribution of the terms $\sim\hbar^0$ (which is in all terms of (2.33)-(2.35) the contribution with the lowest power of $\hbar$). Due to the the support property of R^N the last retarded product vanishes and in the second last we may replace b_μ by $\partial_\mu f$. By using the GLZ relation again we obtain a form to which we can apply the classical current conservation (2.30) and (2.31):

$$\begin{aligned} &= i\,R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}}\otimes\int j^{(0)\,\mu}\,\partial_\mu f,\int g\,\mathcal{L}^{(1)}\Big)|_{\hbar^0}\\ &=\Big[R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}},\int j^{(0)}\,\partial f\Big)\,,\,R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}},\int g\,\mathcal{L}^{(1)}\Big)\Big]_\star|_{\hbar^0}\\ &\quad+i\,R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}}\otimes\int g\,\mathcal{L}^{(1)},\int j^{(0)}\,\partial f\Big)|_{\hbar^0}\\ &=-\Big[R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}},\int \mathcal{L}^{(1)}_1\,\partial g\Big)\,,\,R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}},\int g\,\mathcal{L}^{(1)}\Big)\Big]_\star|_{\hbar^0}\\ &\quad -i\,R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}}\otimes\int g\,\mathcal{L}^{(1)},\int \mathcal{L}^{(1)}_1\,\partial g\Big)|_{\hbar^0}-i\,R^N_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}},\int \mathcal{L}^{(1)}_1\,\partial g\Big)|_{\hbar^0}\;, \end{aligned}\tag{2.34}$$

where $f(x)\,\partial_\nu g(x)=\partial_\nu g(x)$ is taken into account. By means of the GLZ relation the first three retarded products can be expressed by one retarded

product. Applying (A.13) again we end up with

$$= -i\, R^{N\,\mathrm{tree}}_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}} \otimes \int \mathcal{L}^{(1)}_1\,\partial g, \int g\,\mathcal{L}^{(1)}\Big) - i\, R^{N\,\mathrm{tree}}_{S_0}\Big(e_\otimes^{\int g\,\mathcal{L}^{(1)}}, \int \mathcal{L}^{(1)}_1\,\partial g\Big)\ . \tag{2.35}$$

(2.33)=(2.35) is PGI-tree for R^N.

Next we show that PGI-tree is maintained in the transition to the corresponding *connected* time ordered product T^N_c. Following Sect. 5.2 of [8] we define recursively the *connected product*

$$(F_1 \star ... \star F_n)_c \stackrel{\mathrm{def}}{=} (F_1 \star ... \star F_n) - \sum_{|P|\geq 2} \prod_{J\in P} (F_{j_1} \star ... \star F_{j_{|J|}})_c\ , \tag{2.36}$$

where $\{j_1, ..., j_{|J|}\} = J$, $j_1 < ... < j_{|J|}$, the sum runs over all partitions P of $\{1, ..., n\}$ in at least two subsets and $\prod$ means the classical product. Identifying the vertices within each F_j, $(F_1 \star ... \star F_n)_c$ is precisely the contribution of all connected diagrams to $F_1 \star ... \star F_n$. According to Proposition 1 of [9] it holds

$$(F_1 \star ... \star F_n)_c = \mathcal{O}(\hbar^{n-1}) \quad \text{if} \quad F_1, ..., F_n \sim \hbar^0\ . \tag{2.37}$$

Since solely connected diagrams contribute to the retarded products $R_{n,1}$, we conclude from formula (E.6) in [10] that the connected part T^N_c of T^N is obtained from R^N by

$$\begin{aligned} T^N_{n\,c}(F^{\otimes n}) = \sum_{k=1}^{n} i^{k-n} \sum_{l_1+...+l_k=n-k} N(n, k, l_1, ..., l_k) \\ \cdot\big(R^N_{l_1,1}(F^{\otimes l_1}, F) \star ... \star R^N_{l_k,1}(F^{\otimes l_k}, F)\big)_c\ , \end{aligned} \tag{2.38}$$

where $N(n, k, l_1, ..., l_k) \in \mathbb{R}$ is a combinatorical factor. We find $T^N_{n\,c}(F^{\otimes n}) = \mathcal{O}(\hbar^{n-1})$ if $F \sim \hbar^0$. The contribution of the tree diagrams is that part with the lowest power of $\hbar$:

$$T^{N\,\mathrm{tree}}_{n\,c}(F^{\otimes n}) = \sum ...\big(R^{N\,\mathrm{tree}}_{l_1,1}(...) \star ... \star R^{N\,\mathrm{tree}}_{l_k,1}(...)\big)_c|_{\hbar^{n-1}}\ . \tag{2.39}$$

Because d_Q is a graded derivation with respect to the classical and the $\star$-product, it is also a graded derivation with respect to the *connected* product

(2.36). Hence, PGI for $R^{N\,\mathrm{tree}}$ implies

$$d_Q\,T^{N\,\mathrm{tree}}_{n\,c\,S_0}\Big((\int g\,\mathcal{L}^{(1)})^{\otimes n}\Big)=\sum ...\frac{d}{d\lambda}|_{\lambda=0}\Big(R^{N\,\mathrm{tree}}_{l_1,1\,S_0}\Big((\int g\,\mathcal{L}^{(1)}-i\lambda\,\int \mathcal{L}^{(1)}_1\,\partial g)^{\otimes(l_1+1)}\Big)\star ...$$

$$\star R^{N\,\mathrm{tree}}_{l_k,1\,S_0}\Big((\int g\,\mathcal{L}^{(1)}-i\lambda\,\int \mathcal{L}^{(1)}_1\,\partial g)^{\otimes(l_k+1)}\Big)\Big)_c|_{\hbar^0}$$

$$=-i\,n\,T^{N\,\mathrm{tree}}_{n\,c\,S_0}\Big(\int \mathcal{L}^{(1)}_1\,\partial g\otimes(\int g\,\mathcal{L}^{(1)})^{\otimes(n-1)}\Big)\,. \tag{2.40}$$

Finally PGI remains valid also in the step from $T^{N\,\mathrm{tree}}_c$ to $T^{N\,\mathrm{tree}}$, because each tree diagram is the *classical* product of its connected components (the latter are also tree diagrams) and since d_Q is a graded derivation with respect to the classical product. In detail,

$$T^{N\,\mathrm{tree}}_n(F_1\star ...\star F_n)=T^{N\,\mathrm{tree}}_{n\,c}(F_1\star ...\star F_n)+\sum_{|P|\geq 2}\prod_{J\in P}T^{N\,\mathrm{tree}}_{|J|\,c}(F_{j_1}\star ...\star F_{j_{|J|}}), \tag{2.41}$$

where $\prod$, P and J are as in (2.36). With that the statement is obtained analogously to (2.40).

3 From BRS-invariance of the Lagrangian to local conservation of the classical BRS-current

The proof in the preceding Sect. is based on the local BRS-current conservation (2.24) for **classical field theory**. In this Sect. we show that this assumption follows from *BRS-invariance of the Lagrangian for constant coupling*. The latter is verified for concrete models in Sect. 4, in particular for massless spin-2 gauge fields.

In this procedure, solely the Lagrangian and the BRS-transformation s for constant coupling κ need to be given. With that we construct a BRS-transformation $s(g)$ for local coupling $\kappa\,g(x)$ and a corresponding local Noether current $j^\mu(g)$. We show that the divergence of this local BRS-current is indeed of the form (2.24) and in doing so we obtain an explicit formula for the Q-vertex $\mathcal{L}^\mu_1(g)$.

The Lagrangian and the BRS-transformation are assumed to be *formal power series in* κ and we understand all equations in the sense of formal power series. However, we point out that Sects. 3 and 4 are **non-perturbative** classical field theory.

We assume that a BRS-invariant Lagrangian

$$\mathcal{L}_{\rm total} = \sum_{n=0}^{\infty} \kappa^n \, \mathcal{L}^{(n)} =: \mathcal{L}^{(0)} + \mathcal{L}_{\rm int} \ , \quad \mathcal{L}^{(n)} \in \mathcal{P} \ , \tag{3.1}$$

is given, where the free part $\mathcal{L}^{(0)}$ is quadratic in the (derivated) basic fields and higher than first derivatives of the basic fields do not appear in each $\mathcal{L}^{(n)}$. By BRS-invariance we mean that there exists

$$I^\mu = \sum_{n=0}^{\infty} \kappa^n \, I^{(n)\,\mu} \ , \quad I^{(n)\,\mu} \in \mathcal{P} \ , \tag{3.2}$$

such that

$$s \, \mathcal{L}_{\rm total} = -\partial_\mu \, I^\mu \tag{3.3}$$

without using the field equations. We admit BRS-transformations which are formal power series in κ:

$$s = \sum_{n=0}^{\infty} \kappa^n \, s_n \ . \tag{3.4}$$

We assume that s is given on the basic fields φ and that it is extended to a linear map $s : \mathcal{P} \to \mathcal{P}$ (more precisely, to a formal power series of linear maps $s_n : \mathcal{P} \to \mathcal{P}$) by setting

$$s \, (\partial^a \varphi) \stackrel{\rm def}{=} \partial^a (s \, \varphi) \ , \quad a \in \mathbb{N}_0^4 \ , \tag{3.5}$$

for all basic fields φ, and by requiring that s is a graded derivation. It follows

$$s \, (\partial^\mu \, A) = \partial^\mu \, (s \, A) \ , \quad \forall A \in \mathcal{P} \ , \tag{3.6}$$

and that each s_n is a graded derivation which commutes with partial derivatives (3.6). In order that the BRS-symmetry (3.3) can be used to construct the corresponding quantum gauge theory it is needed that s is nilpotent modulo the field equations:

$$s^2 \, (A) \, |_{\mathcal{C}_{S_{\rm total}}} = 0 \ , \quad \forall A \in \mathcal{P} \ , \tag{3.7}$$

where $S_{\rm total} = \int dx \, \mathcal{L}_{\rm total}$. However, in the proof of PGI-tree given in this paper the nil-potency of s is not used.

We first recall the construction of the Noether current for **constant coupling** κ (cf. any book on classical field theory). By using the derivation property of s, (3.6) and the field equations, we get

$$(s\,\mathcal{L}_{\rm total})_{S_{\rm total}} = \Big(\frac{\partial\mathcal{L}_{\rm total}}{\partial\varphi_i}\,s\varphi_i + \frac{\partial\mathcal{L}_{\rm total}}{\partial\varphi_{i,\mu}}\,(s\varphi_i)_{,\mu}\Big)_{S_{\rm total}} = \partial_\mu\Big(\frac{\partial\mathcal{L}_{\rm total}}{\partial\varphi_{i,\mu}}\,s\varphi_i\Big)_{S_{\rm total}}\ , \tag{3.8}$$

where it is summed over all basic fields φ_i. The equality of (3.3) (restricted to $\mathcal{C}_{S_{\rm total}}$) and (3.8) yields that

$$j^\mu \stackrel{\rm def}{=} -\Big(\frac{\partial\mathcal{L}_{\rm total}}{\partial\varphi_{i,\mu}}\,s\varphi_i + I^\mu\Big) = \sum_{n=0}^\infty \kappa^n\,j^{(n)\,\mu}\ , \tag{3.9}$$

is a conserved BRS-current,

$$\partial_\mu\,j^\mu_{S_{\rm total}} = 0\ . \tag{3.10}$$

For later purpose we note

$$j^{(n)\,\mu} = -\Big(\sum_{k=0}^n \frac{\partial\mathcal{L}^{(k)}}{\partial\varphi_{i,\mu}}\,s_{n-k}\,\varphi_i + I^{(n)\,\mu}\Big)\ . \tag{3.11}$$

We are now going to generalize this construction of the BRS-current to **local couplings** $\kappa\,g(x)$, $g\in\mathcal{D}(\mathbb{R}^4)$. Roughly speaking we do this by replacing κ by $\kappa\,g(x)$ everywhere. In detail: from $\mathcal{L}^{(n)}$ (3.1) and $j^{(n)\,\mu}$ (3.11) we construct $\mathcal{L}(g)$ (2.1) and $j^\mu(g)$ (2.2). In the Lagrangian $\mathcal{L}_{\rm total}$ (3.1) we replace the interacting part $\mathcal{L}_{\rm int} = \sum_{\kappa=1}^\infty \kappa^n\mathcal{L}^{(n)}$ by $\mathcal{L}(g)$ (2.1) and we use the same notations S_0, $S(g)$ and $S_{\rm total}(g)$ as in Sect. 2. The *local BRS-transformation* $s(g)$ is also a formal power series in κ,

$$s(g) = \sum_{n=0}^\infty \kappa^n\,s_n(g)\ . \tag{3.12}$$

We determine $s(g)$ by requiring that it is a graded derivation and by its action on the basic fields φ and their partial derivatives:[6]

$$s(g)\,\varphi(x) \stackrel{\rm def}{=} \sum_{n=0}^\infty \kappa^n\,(g(x))^n\,s_n\,\varphi(x)\ , \tag{3.14}$$

[6]Motivated by Sect. 5.2 of [9] an alternative, more explicit definition of $s(g)$ seems to

(where s_n is given from the model with constant coupling (3.4)) and

$$s(g)\,(\partial^a\varphi) \stackrel{\mathrm{def}}{=} \partial^a(s(g)\,\varphi)\ , \quad a\in\mathbb{N}_0^4\ . \tag{3.15}$$

As in (3.6) it follows $s(g)\,(\partial^\mu A) = \partial^\mu\,(s(g)\,A)$, $\forall A\in\mathcal{P}$. Since $s(g)$ is a graded derivation which commutes with partial derivatives, this holds also for

$$s_k(g) \stackrel{\mathrm{def}}{=} \frac{1}{k!}\,\frac{d^k}{d\kappa^k}|_{\kappa=0}\,s(g)\ , \quad k\in\mathbb{N}_0\ . \tag{3.16}$$

For a basic field φ we obtain

$$\begin{aligned} s_k(g)\,\varphi(x) &= (g(x))^k\,s_k\,\varphi(x)\ ,\\ s_k(g)\,\varphi^{,\mu} &= \big(g^k\,s_k\,\varphi\big)^{,\mu} = g^k\,s_k(\varphi^{,\mu}) + g^{,\mu}\,k\,g^{(k-1)}\,s_k\,\varphi\ . \end{aligned} \tag{3.17}$$

$s(g)$ is in general not nilpotent modulo the field equations. But, if $g|_{\mathcal{O}} = 1$ for some region $\mathcal{O}\subset\mathbb{R}^4$, we have $s(g)|_{\mathcal{F}(\mathcal{O})} = s|_{\mathcal{F}(\mathcal{O})}$ (where s is the BRS-transformation of the corresponding model with constant coupling) and hence

$$(s(g))^2\,(F)|_{\mathcal{C}_{S_{\mathrm{total}}(g)}} = 0\ , \quad \forall F\in\mathcal{F}(\mathcal{O})\ . \tag{3.18}$$

As in (3.8) the field equations for $S_{\mathrm{total}}(g)$ imply

$$\begin{aligned} \Big(s(g)\,\mathcal{L}_{\mathrm{total}}(g)\Big)_{S_{\mathrm{total}}(g)} &= \partial_\mu\Big(\frac{\partial\mathcal{L}_{\mathrm{total}}(g)}{\partial\varphi_{i,\mu}}\,s(g)\,\varphi_i\Big)_{S_{\mathrm{total}}(g)}\\ &= \partial_\mu\sum_{n=0}^{\infty}\kappa^n\,g^n\,\sum_{k=0}^{n}\Big(\frac{\partial\mathcal{L}^{(k)}}{\partial\varphi_{i,\mu}}\,s_{n-k}\,\varphi_i\Big)_{S_{\mathrm{total}}(g)}\ . \end{aligned} \tag{3.19}$$

By using the derivation property of $s_l(g)$ and of s_l we obtain

$$\begin{aligned} s_l(g)\,\mathcal{L}^{(k)} &= \frac{\partial\mathcal{L}^{(k)}}{\partial\varphi_i}\,g^l\,s_l\,\varphi_i + \frac{\partial\mathcal{L}^{(k)}}{\partial\varphi_{i,\mu}}\,\big(g^l\,s_l\,(\varphi_{i,\mu}) + g_{,\mu}\,l\,g^{(l-1)}\,s_l\,\varphi_i\big)\\ &= g^l\,s_l\,\mathcal{L}^{(k)} + g_{,\mu}\,l\,g^{(l-1)}\,\frac{\partial\mathcal{L}^{(k)}}{\partial\varphi_{i,\mu}}\,s_l\,\varphi_i\ . \end{aligned} \tag{3.20}$$

---

be natural: namely (3.12) with

$$s_n(g) := \int dx\,(g(x))^n\,\tilde{s}_n(x)\ , \quad \tilde{s}_n(x) := (s_n\varphi_i)(x)\,\frac{\delta}{\delta\varphi_i(x)}\ , \tag{3.13}$$

where it is summed over all basic fields φ_i and s_n is given from the model with constant coupling. Obviously, the so defined $s(g)$ is a graded derivation and one easily verifies that it satisfies (3.14) and (3.15). Hence, this definition (3.13) agrees with the definition given in the main text.

With that we get

$$
s(g)\,\mathcal{L}_{\rm total}(g) = \sum_{n=0}^\infty \kappa^n \sum_{k=0}^n g^k\, s_{n-k}(g)\,\mathcal{L}^{(k)}
$$
$$
= \sum_{n=0}^\infty \kappa^n\, g^n \sum_{k=0}^n s_{n-k}\mathcal{L}^{(k)} + g_{,\mu} \sum_{n=1}^\infty \kappa^n\, g^{(n-1)} \sum_{k=0}^{n-1} (n-k)\, \frac{\partial \mathcal{L}^{(k)}}{\partial \varphi_{i,\mu}}\, s_{n-k}\,\varphi_i \ . \tag{3.21}
$$

With $s = \sum_{n=0}^\infty \kappa^n\, s_n$ and (3.3) the first term is equal to

$$
\sum_{n=0}^\infty \kappa^n\, g^n\, \big(s\,\mathcal{L}_{\rm total}\big)^{(n)} = -\sum_{n=0}^\infty \kappa^n\, g^n\, \partial_\mu\, I^{(n)\,\mu}
$$
$$
= -\partial_\mu \sum_{n=0}^\infty \kappa^n\, g^n\, I^{(n)\,\mu} + g_{,\mu} \sum_{n=1}^\infty \kappa^n\, n\, g^{(n-1)}\, I^{(n)\,\mu} \ . \tag{3.22}
$$

We insert (3.22) into (3.21) and the resulting equation into the left side of (3.19), and then we use (3.11). This yields the local current conservation (2.24), where $\mathcal{L}_1^\mu(g)$ is given by (2.3) and

$$
\mathcal{L}_1^{(n)\,\mu} \stackrel{\rm def}{=} -\Big(\sum_{k=0}^{n-1} (n-k)\, \frac{\partial \mathcal{L}^{(k)}}{\partial \varphi_{i,\mu}}\, s_{n-k}\,\varphi_i + n\, I^{(n)\,\mu}\Big)\ , \quad n = 1, 2, \dots \ . \tag{3.23}
$$

With that the proof of PGI-tree is complete for models with a BRS-invariant Lagrangian (3.3).

d_Q can be viewed as a graded derivation $d_Q : \mathcal{P}|_{\mathcal{C}_{S_0}} \to \mathcal{P}|_{\mathcal{C}_{S_0}}$. Interpreting d_Q in this sense we would like to prove

$$
d_Q\, A_{S_0} = i\,(s_0\, A)_{S_0}\ , \quad \forall A \in \mathcal{P} \ . \tag{3.24}
$$

Since d_Q and s_0 are both graded derivations which commute with partial derivatives it suffices to prove (3.24) for A running through all basic fields φ_i. In models with solely massless fields it usually holds that $s_0\,\varphi_i$ is a divergence, i.e.

$$
s_0\,\varphi_i = \partial_\mu\, \phi_i^\mu \quad \text{for some} \quad \phi_i^\mu \in \mathcal{P} \ , \tag{3.25}
$$

for all basic fields φ_i. If this holds true, (3.24) follows from part (i) of the following Corollary of PGI-tree.

Corollary 2. *Let a free Lagrangian $\mathcal{L}^{(0)}$ be given which is invariant with respect to a free BRS-transformation s_0,*

$$s_0\,\mathcal{L}^{(0)} = -\partial_\mu\, I^{(0)\,\mu} \quad \textit{for some} \quad I^{(0)\,\mu} \in \mathcal{P}\ . \tag{3.26}$$

In terms of the corresponding Noether current $j^{(0)\,\mu}$ (3.11) we define d_Q by (2.10)-(2.11).

(i) If $s_0\, A = \partial_\mu\, B^\mu$ for some B^μ, then A satisfies the relation (3.24).

(ii) For an arbitrary $P \in \mathcal{P}$ it holds

$$\partial^\nu \left(d_Q\, P_{S_0} - i\,(s_0\, P)_{S_0}\right) = 0\ . \tag{3.27}$$

Proof. (i) To the given free Lagrangian we add the interaction $\mathcal{L}_{\rm int} := \kappa\, A$ and choose $s := s_0$. This model is BRS invariant:

$$s\mathcal{L}_{\rm total} = s_0\,\mathcal{L}^{(0)} + \kappa\, s_0\, A = -\partial_\mu (I^{(0)\,\mu} - \kappa\, B^\mu)\ . \tag{3.28}$$

So, our assumption (3.3) is satisfied and, hence, PGI-tree holds with

$$\mathcal{L}_1^{(1)\,\mu} = -I^{(1)\,\mu} = B^\mu\ , \tag{3.29}$$

where (3.23) is used. To first order PGI-tree reads

$$d_Q\, A_{S_0} = d_Q\, \mathcal{L}^{(1)}_{S_0} = i\,\partial_\mu\, \mathcal{L}^{(1)\,\mu}_{1\ S_0} = i\,\partial_\mu\, B^\mu_{S_0} = i\,(s_0\, A)_{S_0}\ . \tag{3.30}$$

(ii) Since $s_0\,(\partial^\nu P) = \partial_\mu\,(\eta^{\mu\nu}\, s_0\, P)$, part (i) applies to $A = \partial^\nu P$. $\square$

Remarks: (1) From (3.24) and PGI-tree it follows

$$i\,(s_0\,\mathcal{L}^{(1)})_{S_0} = d_Q\,\mathcal{L}^{(1)}_{S_0} = i\,\partial_\nu \mathcal{L}^{(1)\,\nu}_{1\,S_0}\ . \tag{3.31}$$

The relation

$$(s_0\,\mathcal{L}^{(1)})_{S_0} = \partial_\nu \mathcal{L}^{(1)\,\nu}_{1\,S_0} \tag{3.32}$$

(where $\mathcal{L}_1^{(1)\,\mu}$ is given by $\mathcal{L}_1^{(1)\,\mu} = -\frac{\partial \mathcal{L}^{(0)}}{\partial \varphi_{i,\mu}}\, s_1\,\varphi_i - I^{(1)\,\mu}$ (3.23)) can be verified directly, i.e. without using PGI-tree. Namely, by using $-\partial_\mu\, I^{(1)\,\mu} = s_0\,\mathcal{L}^{(1)} +$

$s_1\,\mathcal{L}^{(0)}$ (3.3), the derivation property of s_1 and the free field equations we obtain

$$\big(s_0\,\mathcal{L}^{(1)}-\partial_\mu\mathcal{L}_1^{(1)\,\mu}\big)_{S_0}=\Big(-s_1\,\mathcal{L}^{(0)}+\partial_\mu\Big(\frac{\partial\mathcal{L}^{(0)}}{\partial\varphi_{i,\mu}}\,s_1\,\varphi_i\Big)\Big)_{S_0}$$
$$=\Big(-\frac{\partial\mathcal{L}^{(0)}}{\partial\varphi_i}\,s_1\,\varphi_i-\frac{\partial\mathcal{L}^{(0)}}{\partial\varphi_{i,\mu}}\,(s_1\,\varphi_i)_{,\mu}+\partial_\mu\Big(\frac{\partial\mathcal{L}^{(0)}}{\partial\varphi_{i,\mu}}\,s_1\,\varphi_i\Big)\Big)_{S_0}$$
$$=\Big(\partial_\mu\frac{\partial\mathcal{L}^{(0)}}{\partial\varphi_{i,\mu}}-\frac{\partial\mathcal{L}^{(0)}}{\partial\varphi_i}\Big)_{S_0}\,(s_1\,\varphi_i)_{S_0}=0\ . \tag{3.33}$$

(2) If higher (≥ 2) derivatives of the basic fields appear in $\mathcal{L}_{\rm total}$ (3.1) the construction of a conserved BRS-current is still possible in the case of constant coupling: the field equations

$$\sum_{l\in\mathbb{N}_0}(-1)^l\,\partial_{\mu_1}...\partial_{\mu_l}\,\frac{\partial\,\mathcal{L}_{\rm total}}{\partial\,\varphi_{i,\mu_1...\mu_l}}=0 \tag{3.34}$$

imply that the current

$$j^\mu\stackrel{\rm def}{=}-\Big(\sum_{l\in\mathbb{N}_0}\sum_{j=0}^{l}(-1)^j\,\partial_{\mu_1}...\partial_{\mu_j}\Big(\frac{\partial\,\mathcal{L}_{\rm total}}{\partial\,\varphi_{i,\mu\mu_1...\mu_l}}\Big)\,s\varphi_{i,\mu_{j+1}...\mu_l}+I^\mu\Big) \tag{3.35}$$

is conserved. But in case of a local coupling terms $\sim g_{,\mu\mu_1}(x),\sim g_{,\mu}(x)\,g_{,\mu_1}(x),...$ appear in (3.19)-(3.21) and, hence, also in the divergence of the local BRS-current, i.e. on the right side of (2.24).

4 Models

4.1 Massless Yang-Mills theories

To point out the similarity of the BRS-symmetry for massless spin-1 and spin-2 gauge fields we first recall the spin-1 case. In terms of the covariant derivative

$$D_{ab}^\mu\stackrel{\rm def}{=}\delta_{ab}\partial^\mu-\kappa\,f_{abc}\,A_c^\mu\ , \tag{4.1}$$

(where f_{abc} is totally antisymmetric and satisfies the Jacobi identity) the BRS-transformation reads

$$s\,A_a^\mu=D_{ab}^\mu\,u_b\ ,\quad s\,u_a=-\frac{\kappa}{2}\,f_{abc}\,u_b\,u_c\ ,\quad s\tilde{u}_a=-\partial_\mu\,A_a^\mu\ . \tag{4.2}$$

Due to

$$F_a^{\mu\nu} \equiv \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + \kappa\, g\, f_{abc} A_b^\mu\, A_c^\nu \tag{4.3}$$

the Yang-Mills Lagrangian

$$\mathcal{L}_{\rm YM} = -\frac{1}{4}\, F_a^{\mu\nu}\, F_{a\,\mu\nu} \tag{4.4}$$

is of the form $\mathcal{L}_{\rm YM} = \mathcal{L}_{\rm YM}^{(0)} + \kappa\, \mathcal{L}_{\rm YM}^{(1)} + \kappa^2\, \mathcal{L}_{\rm YM}^{(2)}$. The gauge fixing Lagrangian is of zeroth order in κ,

$$\mathcal{L}_{\rm GF} = -\frac{1}{2}\, (\partial_\nu A_a^\nu)^2\ , \tag{4.5}$$

where we choose Feynman gauge. However, the ghost Lagrangian has also a term linear in κ,

$$\mathcal{L}_{\rm ghost} = \partial_\mu \tilde{u}_a\, s\, A_a^\mu\ . \tag{4.6}$$

Note $s_k = 0, \forall k \geq 2$, and $\mathcal{L}_{\rm total}^{(j)} = 0, \forall j \geq 3$, where $\mathcal{L}_{\rm total} \stackrel{\rm def}{=} \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm ghost}$. The BRS-transformation is nilpotent modulo the field equations,[7]

$$s^2\, A_a^\mu = 0\ , \quad s^2\, u_a = 0\,, \quad s^2 \tilde{u}_a = \frac{\delta\, S_{\rm total}}{\delta\, \tilde{u}_a}\ , \tag{4.7}$$

with $S_{\rm total} \stackrel{\rm def}{=} \int dx\, \mathcal{L}_{\rm total}$. (Since s is a graded derivation which commutes with partial derivatives the vanishing of s^2 on the basic fields implies $s^2 = 0$.)

We are now going to verify the BRS-invariance of $\mathcal{L}_{\rm total}$. $s\,\mathcal{L}_{\rm YM}$ vanishes, since the BRS-transformation of A_a^μ has the form of an infinitesimal gauge transformation and since the gauge variation of $\mathcal{L}_{\rm YM}$ vanishes. For $\mathcal{L}_{\rm GF}$ and $\mathcal{L}_{\rm ghost}$ we obtain

$$\begin{aligned} s\,(\mathcal{L}_{\rm GF} + \mathcal{L}_{\rm ghost}) &= -(\partial_\nu A_a^\nu)\, \partial_\mu (s\, A_a^\mu) - (\partial_\mu \partial_\nu A_a^\nu)\, s\, A_a^\mu - \partial_\mu \tilde{u}_a\, s^2 A_a^\mu \\ &= -\partial_\mu \big((\partial_\nu A_a^\nu)\, D_{ab}^\mu\, u_b\big) =: -\partial_\mu\, I^\mu\ . \end{aligned} \tag{4.8}$$

Our formula (3.23) yields the following explicit expressions for the Q-vertex, which agree with the literature [14, 27, 12]:

$$\mathcal{L}_1^{(1)\,\nu} = f_{abc} [A_{a\,\mu}\, u_b\, (\partial^\nu A_c^\mu - \partial^\mu A_c^\nu) - \frac{1}{2} u_a\, u_b\, \partial^\nu \tilde{u}_c]\ , \tag{4.9}$$

$$\mathcal{L}_1^{(2)\,\nu} = f_{abr} f_{cdr}\, A_{a\,\mu}\, u_b\, A_c^\nu\, A_d^\mu\ , \tag{4.10}$$

$$\mathcal{L}_1^{(j)\,\nu} = 0\ , \quad \forall j \geq 3\ . \tag{4.11}$$

[7] If one introduces the Nakanishi-Lautrup fields B_a [24], the BRS transformation s is nilpotent in $\mathcal{P}$ (i.e. without using the field equations). s is then modified as follows: $s\,\tilde{u}_a = -B_a$ and $s\, B_a = 0$.

For the zeroth order of the BRS-current our expression (3.11) gives

$$\begin{aligned} j^{(0)\,\mu} &= \partial_\tau A_a^\tau\, \partial^\mu u_a - (\partial^\mu \partial_\tau A_a^\tau)\, u_a \\ &-\partial_\nu \left((A_a^{\mu,\nu} - A_a^{\nu,\mu})\, u_a \right) + (\Box A_a^\mu)\, u_a \ . \end{aligned} \tag{4.12}$$

The terms in the second line do not contribute to d_Q (2.13)-(2.14). This is obvious for the last term due to $\Box A^\mu_{a\,S_0} = 0$. Turning to the second last term we point out that generally a term of the form $\partial_\nu(K^{\mu\nu} - K^{\nu\mu})$ does not contribute to d_Q (2.14):

$$\int_{x^0=\mathrm{const.}} d^3x\, \partial_l^x\, [(K^{0l}_{S_0}(x^0,\vec{x}) - K^{l0}_{S_0}(x^0,\vec{x})), F_{S_0}]_\star^\mp = 0 \ . \tag{4.13}$$

Hence, d_Q can be constructed from the free BRS-current $(\partial A)\,\partial^\mu u - (\partial^\mu \partial A)\,u$ as it is done in [14, 27, 12].

4.2 Massive spin-1 fields

It is instructive to see how the BRS-formalism of the preceding Subsect. is modified for *massive* fields. To simplify the notations we consider the most simple non-Abelian model: the $SU(2)$ Higgs-Kibble model, which describes three spin-1 fields, $A_a^\mu, a = 1, 2, 3$, and the bosonic scalar fields form a complex $SU(2)$ doublet,

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_2 + i\phi_1 \\ v + H - i\phi_3 \end{pmatrix} \ . \tag{4.14}$$

The shift $v \in \mathbb{R}_+$ will be chosen such that the Higgs potential has a non-trivial minimum at $\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$ (4.17), as usual in the Higgs mechanism. The corresponding covariant derivative is

$$\mathcal{D}^\mu = (\mathbf{1}\, \partial^\mu - \kappa\, \frac{i}{2}\, A_a^\mu\, \sigma_a) \ , \tag{4.15}$$

where $(\sigma_a)_{a=1,2,3}$ are the Pauli matrices. Requiring renormalizability and $SU(2)$-invariance the Higgs Lagrangian takes the form

$$\mathcal{L}_\Phi = (\mathcal{D}_\mu \Phi)^+ (\mathcal{D}^\mu \Phi) + \mu^2\, \Phi^+ \Phi - \lambda\, (\Phi^+ \Phi)^2 \ , \tag{4.16}$$

Choosing $\mu^2 > 0$ and $\lambda > 0$ one obtains

$$v^2 = \frac{\mu^2}{\lambda} \ . \tag{4.17}$$

The $SU(2)$-invariant Yang-Mills Lagrangian $\mathcal{L}_{\mathrm{YM}}$ is still given by (4.3)-(4.4). By the Higgs mechanism the three spin-1 fields A_a^μ $(a = 1, 2, 3)$ get the same mass

$$m = \frac{\kappa\, v}{2} > 0 \; . \tag{4.18}$$

From the quantization of spin-1 fields one knows that in the massive case the condition $\partial_\mu A_a^\mu = 0$ on observables is replaced by $(\partial_\mu A_a^\mu + m\phi_a) = 0$. (For free fields this is explained in Sect. 3 of [11].) Therefore,

$$s\,\tilde{u}_a = -(\partial_\mu A_a^\mu + m\phi_a) \; , \tag{4.19}$$

and this replacement appears also in the gauge fixing term

$$\mathcal{L}_{\mathrm{GF}} = -(\partial_\mu A_a^\mu + m\phi_a)^2 = \mathcal{L}_{\mathrm{GF}}^{(0)} \; , \tag{4.20}$$

where we choose again Feynman gauge. The BRS-transformation of A_a^μ and $(\phi_a,\, H)$ has the form of an infinitesimal gauge transformation of the *unshifted* fields. That is, $s\,A_a^\mu = D_{ab}^\mu\, u_b$ remains unchanged and

$$s\,\Phi = \frac{\kappa\, i}{2}\,\sigma_a\, u_a\, \Phi \; . \tag{4.21}$$

The latter reads explicitly

$$\begin{aligned} s\,\phi_a &= m\, u_a + \frac{\kappa}{2}\,(f_{abc}\,\phi_b\, u_c + H\, u_a) \; , \\ s\,H &= -\frac{\kappa}{2}\,\phi_a\, u_a \; . \end{aligned} \tag{4.22}$$

It follows

$$s\,\mathcal{L}_{\mathrm{YM}} = 0 \; , \quad s\,\mathcal{L}_{\Phi} = 0 \; . \tag{4.23}$$

To keep $s^2\, A_a^\mu = 0 = s^2\, u_a$, the BRS-transformation of u_a is unchanged: $s\, u_a = -\frac{\kappa}{2}\, f_{abc}\, u_b\, u_c$. With that one easily verifies $s^2\,\phi_a = 0 = s^2\, H$:

$$s^2\,\Phi = \frac{\kappa\, i}{2}\,\sigma_a\,(su_a)\,\Phi - \frac{\kappa\, i}{2}\,\sigma_a\, u_a\, s\Phi = 0 \; , \tag{4.24}$$

due to $(\sigma_a\, u_a)\,(\sigma_b\, u_b) = i\, f_{abc}\, u_a\, u_b\, \sigma_c$.

The ghost Lagrangian is chosen such that $s\,(\mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{ghost}})$ is a divergence: generalizing (4.6) and (4.8) one sets

$$\mathcal{L}_{\mathrm{ghost}} \stackrel{\mathrm{def}}{=} \partial_\mu \tilde{u}_a\; s\, A_a^\mu - m\,\tilde{u}_a\, s\,\phi_a = \mathcal{L}_{\mathrm{ghost}}^{(0)} + \kappa \mathcal{L}_{\mathrm{ghost}}^{(1)} \; , \tag{4.25}$$

which yields indeed

$$s\,(\mathcal{L}_{\rm GF}+\mathcal{L}_{\rm ghost})=-\partial_\mu\, I^\mu\ ,\quad I^\mu\stackrel{\rm def}{=}(\partial_\nu A^\nu_a+m\phi_a)\,D^\mu_{ab}u_b=I^{(0)\,\mu}+\kappa\, I^{(1)\,\mu}\ . \tag{4.26}$$

We end up with the total Lagrangian

$$\mathcal{L}_{\rm total}\stackrel{\rm def}{=}\mathcal{L}_{\rm YM}+\mathcal{L}_\Phi+\mathcal{L}_{\rm GF}+\mathcal{L}_{\rm ghost}=-\frac{\lambda\,v^4}{4}+\mathcal{L}^{(0)}+\kappa\,\mathcal{L}^{(1)}+\kappa^2\,\mathcal{L}^{(2)}\ . \tag{4.27}$$

The constant $-\frac{\lambda\,v^4}{4}$ is irrelevant and all terms of $\mathcal{L}^{(2)}$ come from $\mathcal{L}_{\rm YM}+\mathcal{L}_\Phi$. Note that $\mathcal{L}^{(0)}$ contains a divergence term, $-m\,\partial_\mu(A^\mu_a\phi_a)$, which is irrelevant for the field equations but contributes to $j^{(n)\,\mu}$ (3.11) and $\mathcal{L}^{(n)\,\mu}$ (3.23). For later purpose we give $\mathcal{L}^{(0)}$ explicitly:

$$\begin{aligned}\mathcal{L}^{(0)}=&-\frac{1}{4}(A^{\mu,\nu}_a-A^{\nu,\mu}_a)(A_{a\,\mu,\nu}-A_{a\,\nu,\mu})-\frac{1}{2}(\partial A_a)^2+\frac{m^2}{2}\,A^\mu_aA_{a\,\mu}\\&+\partial_\mu\tilde{u}_a\,\partial^\mu u_a-m^2\,\tilde{u}_au_a\\&+\frac{1}{2}\,\partial_\mu\phi_a\,\partial^\mu\phi_a-\frac{m^2}{2}\,\phi_a^2+\frac{1}{2}\,\partial_\mu H\,\partial^\mu H-\lambda\,v^2\,H^2\\&-m\,\partial_\mu(A^\mu_a\,\phi_a)\ .\end{aligned} \tag{4.28}$$

As in the massless case it is only the vanishing of $s^2\tilde{u}_a$ which relies on the field equations:

$$s^2\,A^\mu_a=0\ ,\quad s^2\,u_a=0\,,\quad s^2\tilde{u}_a=\frac{\delta\,S_{\rm total}}{\delta\,\tilde{u}_a}\ ,\quad s^2\,\phi_a=0\ ,\quad s^2\,H=0 \tag{4.29}$$

and, hence, footnote 7 applies also to the massive case.

Inserting (4.26) and (4.28) into our formulas (3.11) (BRS-current) and (3.23) (Q-vertex) we obtain

$$\begin{aligned}j^{(0)\,\mu}=&(\partial_\tau A^\tau_a+m\,\phi_a)\,\partial^\mu u_a-(\partial^\mu(\partial_\tau A^\tau_a+m\,\phi_a))\,u_a\\&-\partial_\nu\left((A^{\mu,\nu}_a-A^{\nu,\mu}_a)\,u_a\right)+((\square+m^2)A^\mu_a)\,u_a\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}\mathcal{L}^{(1)\,\mu}_1=&f_{abc}[A_{a\,\nu}\,u_b\,(\partial^\mu A^\nu_c-\partial^\nu A^\mu_c)-\frac{1}{2}u_a\,u_b\,\partial^\mu\tilde{u}_c+\frac{m}{2}\,A^\mu_a\,\phi_b\,u_c-\frac{1}{2}\,\partial^\mu\phi_a\,\phi_b\,u_c]\\&+\frac{1}{2}\,\partial^\mu H\,u_a\,\phi_a-\frac{1}{2}\,H\,u_a\,\partial^\mu\phi_a+\frac{m}{2}\,A^\mu_a\,u_a\,H\end{aligned} \tag{4.31}$$

and again $\mathcal{L}_1^{(k)\,\nu} = 0$, $\forall k \geq 3$. For the same reason as in the massless case (4.13) the terms in the second line of (4.30) do not contribute to d_Q. With that our results (4.30) and (4.31) agree with the literature [15, 12].

Remarks: (1) If one omits the divergence $-m\,\partial_\mu(A_a^\mu \phi_a)$ in $\mathcal{L}^{(0)}$ an additional term appears in $s\,\mathcal{L}_{\rm total}$: I^μ is replaced by $(I^\mu - m\,s\,(A_a^\mu\phi_a))$. With that our results for $j^{(0)}$ (4.30) and $\mathcal{L}_1^{(1)}$ (4.31) remain unchanged.
(2) For massive spin-1 fields $s_0\varphi_l$ is not a divergence (3.25) for the basic fields $\varphi_l = \tilde{u}$, ϕ. However, since the free field equations are differential equations of *second* order, we obtain $i\,(s_0\varphi_l)_{S_0} = d_Q\,\varphi_{l\,S_0}$, $\varphi_l = \tilde{u}$, ϕ, from part (ii) of Corollary 2.

4.3 Massless spin-2 gauge fields

In this Subsect. we complete our proof of PGI-tree for massless spin-2 gauge fields: we show that classical gravity can be formulated by a BRS-invariant Lagrangian $\mathcal{L}_{\rm total}$ (3.3) which fits in our formalism. To satisfy the latter, $\mathcal{L}_{\rm total}$ must be a (formal) power series in κ and it must be a polynomial only in zeroth and first derivatives of the basic fields. (It will turn out that both properties are non-trivial.) Such a BRS-formulation of gravity was given by Kugo and Ojima in Sect. 2 of [23]. In that formalism we choose Feynman gauge $\alpha_0 = 1$ and eliminate the Nakanishi-Lautrup field b_μ [24] (by inserting the field equation for b_μ). In view of perturbation theory around the Minkowski metric $\eta^{\mu\nu}$ we introduce a field $h^{\mu\nu}$ which is the deviation from $\eta^{\mu\nu}$, in terms Goldberg variables $\tilde{g}^{\mu\nu}$ (for details see e.g. [27]):

$$\tilde{g}^{\mu\nu}(x) \stackrel{\rm def}{=} \sqrt{-g(x)}\, g^{\mu\nu}(x) = \eta^{\mu\nu} + \kappa\, h^{\mu\nu}(x)\ . \tag{4.32}$$

The inverse tensor is a kind of geometric series in κh (which we understand as formal power series in κ):

$$\tilde{g}_{\mu\nu} \stackrel{\rm def}{=} \frac{1}{\sqrt{-g}}\, g_{\mu\nu} = \eta_{\mu\nu} - \kappa\, h_{\mu\nu} + \kappa^2\, h_{\mu\alpha} h^\alpha_{\ \nu} - \ldots\ , \tag{4.33}$$

where

$$h^\alpha_{\ \nu} = \eta_{\nu\tau}\, h^{\alpha\tau}\ , \quad h_{\mu\nu} = \eta_{\mu\rho}\,\eta_{\nu\tau}\, h^{\rho\tau}\ , \tag{4.34}$$

and we set $h \stackrel{\rm def}{=} h^\mu_\mu$. The field algebra $\mathcal{P}$ is the polynomial algebra generated by $h^{\mu\nu}$, the fermionic vector ghost fields u^μ, $\tilde{u}_\mu$ and all partial derivatives of these fields. (The indices of u and $\tilde{u}$ are also raised and lowered by means

of η.) According to [23] the BRS-transformation is of the form $s = s_0 + \kappa\, s_1$ and on the basic fields it is given by

$$\begin{aligned} s\, h^{\mu\nu} &= u^{\nu,\mu} + u^{\mu,\nu} - \eta^{\mu\nu}\, u^\rho_{,\rho} + \kappa\, (h^{\mu\sigma}\, u^\nu_{,\sigma} + h^{\nu\sigma}\, u^\mu_{,\sigma} - (h^{\mu\nu}\, u^\rho)_{,\rho})\ , \\ s\, u^\mu &= -\kappa\, (u^\lambda\, u^\mu_{,\lambda})\ , \\ s\, \tilde{u}_\mu &= -h_{\mu\rho}^{\ ,\rho}\ . \end{aligned} \tag{4.35}$$

(The sign of $s\, u^\mu$ is determined by the requirement $s^2\, h^{\mu\nu} = 0$.) By using

$$D^{\mu\nu}_\rho \stackrel{\text{def}}{=} (\eta^{\mu\sigma} + \kappa\, h^{\mu\sigma})\, \delta^\nu_\rho\, \partial_\sigma + (\eta^{\nu\sigma} + \kappa\, h^{\nu\sigma})\, \delta^\mu_\rho\, \partial_\sigma - \partial_\rho\big((\eta^{\mu\nu} + \kappa\, h^{\mu\nu})\ \cdot\ \big) \tag{4.36}$$

we may also write

$$s\, h^{\mu\nu} = D^{\mu\nu}_\rho\, u^\rho\ . \tag{4.37}$$

In terms of the Christoffel symbols

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\mu}\, (g_{\beta\mu,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) \tag{4.38}$$

the Einstein-Hilbert Lagrangian reads

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{\kappa^2}\, \sqrt{-g}\, g^{\mu\nu}\, (\Gamma^\rho_{\nu\rho,\mu} - \Gamma^\rho_{\mu\nu,\rho} + \Gamma^\rho_{\mu\sigma}\, \Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\mu\nu}\, \Gamma^\rho_{\sigma\rho})\ . \tag{4.39}$$

But $\mathcal{L}_{\mathrm{E}}$ is unsuitable for our formalism: it contains terms $\sim \kappa^{-1}$ and second derivatives of $h^{\mu\nu}$. Both shortcomings can be removed by subtracting a divergence [23]:

$$\mathcal{L}'_{\mathrm{E}} \stackrel{\text{def}}{=} \mathcal{L}_{\mathrm{E}} - \partial_\mu \mathcal{D}^\mu\ , \tag{4.40}$$

$$\begin{aligned} \mathcal{D}^\mu &\stackrel{\text{def}}{=} \frac{1}{\kappa^2}\, (\tilde{g}^{\mu\nu}\, \Gamma^\lambda_{\nu\lambda} - \tilde{g}^{\rho\sigma}\, \Gamma^\mu_{\rho\sigma}) = \frac{1}{\kappa^2}\, \big(\frac{1}{2}\, \tilde{g}^{\mu\nu}\, \tilde{g}_{\alpha\beta}\, \partial_\nu \tilde{g}^{\alpha\beta} + \partial_\nu \tilde{g}^{\mu\nu}\big) \\ &= \frac{1}{\kappa}\, \mathcal{D}^{(-1)\,\mu} + \mathcal{O}(\kappa^0)\ , \quad \mathcal{D}^{(-1)\,\mu} = \frac{1}{2}\, h^{,\mu} + h^{\mu\nu}_{,\nu}\ . \end{aligned} \tag{4.41}$$

By inserting (4.32)-(4.33) and (4.38) we indeed obtain a formal power series in κ for $\mathcal{L}'_{\mathrm{E}}$:

$$\mathcal{L}'_{\mathrm{E}} = \sum_{n=0}^{\infty} \kappa^n\, \mathcal{L}'^{(n)}_{\mathrm{E}}\ , \tag{4.42}$$

see Sect. 5.5 of [27]. The total Lagragian $\mathcal{L}_{\text{total}}$ of the BRS-formalism is obtained by adding a gauge fixing term $\mathcal{L}_{\text{GF}}$ and a ghost term $\mathcal{L}_{\text{ghost}}$:

$$\mathcal{L}_{\text{total}} = \mathcal{L}'_{\text{E}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{ghost}} = \sum_{n=0}^{\infty} \kappa^n \mathcal{L}^{(n)} , \tag{4.43}$$

$$\mathcal{L}_{\text{GF}} = \frac{1}{2} h^{\alpha\beta}_{\ ,\beta}\, h_{\alpha\rho}^{\ ,\rho} = \mathcal{L}^{(0)}_{\text{GF}} , \tag{4.44}$$

$$\mathcal{L}_{\text{ghost}} = -\frac{1}{2} (\tilde{u}_{\nu,\mu} + \tilde{u}_{\mu,\nu})\, s\, h^{\mu\nu} = \mathcal{L}^{(0)}_{\text{ghost}} + \kappa\, \mathcal{L}^{(1)}_{\text{ghost}} . \tag{4.45}$$

With that the BRS-transformation is nilpotent modulo the field equations:

$$s^2\, h^{\mu\nu} = 0 , \quad s^2\, u^{\mu} = 0 , \quad s^2\, \tilde{u}_{\mu} = -\eta_{\mu\tau}\, (D^{\tau\nu}_{\rho}\, u^{\rho})_{,\nu} = -\frac{\delta\, S_{\text{total}}}{\delta\, \tilde{u}^{\mu}} . \tag{4.46}$$

We turn to the verification of the BRS-invariance of $\mathcal{L}_{\text{total}}$. Again, the BRS-transformation of the gauge field $h^{\mu\nu}$ has the form of an infinitesimal gauge transformation (i.e. general coordinate transformation) and, hence, $s\, \mathcal{L}_{\text{E}}$ is known from the gauge variation of $\mathcal{L}_{\text{E}}$ [23]:

$$s\, \mathcal{L}_{\text{E}} = -\partial_{\mu}(\kappa\, \mathcal{L}_{\text{E}}\, u^{\mu}) . \tag{4.47}$$

By the same calculation as in (4.8) we obtain

$$s\, (\mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{ghost}}) = \partial_{\mu} F^{\mu} , \quad F^{\mu} \stackrel{\text{def}}{=} h_{\nu\rho}^{\ ;\rho}\, D^{\mu\nu}_{\lambda}\, u^{\lambda} = F^{(0)\,\mu} + \kappa\, F^{(1)\,\mu} . \tag{4.48}$$

Summing up we get

$$s\, \mathcal{L}_{\text{total}} = -\partial_{\mu}\, I^{\mu} , \quad I^{\mu} \stackrel{\text{def}}{=} \kappa\, \mathcal{L}_{\text{E}}\, u^{\mu} + s\, \mathcal{D}^{\mu} - F^{\mu} + \frac{1}{\kappa}\, \partial_{\rho}(u^{\rho,\mu} - u^{\mu,\rho}) = \sum_{n=0}^{\infty} \kappa^n\, I^{(n)\,\mu} . \tag{4.49}$$

In I^{μ} we have added the conserved vector field $\frac{1}{\kappa}\, \partial_{\rho}(u^{\rho,\mu} - u^{\mu,\rho})$ to cancel $s_0\, \mathcal{D}^{(-1)\,\mu} = \Box u^{\mu} - u^{\rho\ ,\mu}_{\ ,\rho}$, because in our formalism I^{μ} is assumed to be a formal power series in κ.

By means of our formula (3.11) we compute the zeroth order of the BRS-current. We obtain

$$j^{(0)\,\mu}_{S_0} = -h^{\alpha\beta}_{S_0\,,\beta}\, \partial^{\mu} u_{S_0\,,\alpha} + (\partial^{\mu} h^{\alpha\beta}_{S_0\,,\beta})\, u_{S_0\,,\alpha} + \partial_{\rho}(K^{\mu\rho}_{S_0} - K^{\rho\mu}_{S_0}) , \tag{4.50}$$

where

$$K^{\rho\mu} \stackrel{\text{def}}{=} \frac{1}{2}\, h^{,\rho}\, u^{\mu} + h^{\alpha\rho,\mu}\, u_{\alpha} + h^{\rho\nu}_{\ ,\nu}\, u^{\mu} + h^{\rho\nu}\, u^{\mu}_{\ ,\nu} , \tag{4.51}$$

for details see Appendix B. As explained in (4.13), the term $\partial_{\rho}(K^{\mu\rho} - K^{\rho\mu})$ does not contribute to d_Q. The other terms are precisely the terms which are used for the construction of d_Q in [29, 27, 18].

5 Outlook

What do we learn from this paper with respect to model building, i.e. the task:

- given a free BRS invariant theory $(\mathcal{L}^{(0)}, s_0)$ which satisfies (2.17), find a non-trivial deformation $\mathcal{L}^{(0)} \to \mathcal{L}_{\text{total}} = \mathcal{L}^{(0)} + \mathcal{L}_{\text{int}}$ (3.1) such that $\mathcal{L}_{\text{total}}$ satisfies PGI-tree (and some obvious further conditions, e.g. Lorentz invariance, $\mathcal{L}^*_{\text{total}} = \mathcal{L}_{\text{total}}$ and ghost number zero)?

In the literature this problem is usually treated by making a polynomial ansatz for $\mathcal{L}_{\text{int}}$ and working out the condition of PGI-tree, as mentioned in point (B) of the introduction. Due to this paper one can proceed alternatively, namely one searches for deformations $\mathcal{L}^{(0)} \to \mathcal{L}_{\text{total}}$ and $s_0 \to s$ (with (3.4), (3.5) and (3.7)) such that $s\mathcal{L}_{\text{total}}$ is a divergence (3.3). This amounts to an inductive determination of the sequences $(\mathcal{L}^{(n)})_n$ and $(s_n)_n$. (In Sect. 5.3 and Appendix B of [9] and in [21] analogous procedures are given to solve the local current conservation (2.24).) It is not yet investigated, whether this procedure yields the *most general* solution of the above task, but it would be rather surprising if there were additional solutions.

The method of proof given in this paper applies not only to BRS-symmetry; all Lagrangians $\mathcal{L}_{\text{total}}$ and infinitesimal symmetry transformations s which satisfy (3.1)-(3.5) and (2.17) (where d_Q is constructed by (2.13) from the zeroth order $j^{(0)\,\mu}$ (3.11) of the Noether current belonging to $(\mathcal{L}_{\text{total}}, s)$) are admitted. Since the nilpotency (3.7) of the BRS-transformation is not used in our proof, the restrictions on s are not strong. A trivial example are global $U(1)$-symmetries, e.g. charge conservation in QED: for the spinors ψ and $\bar{\psi}$ let $s\psi = i\psi$, $s\bar{\psi} = -i\bar{\psi}$, and the photon field A^μ is uncharged, $sA^\mu = 0$. The s-variation of the Lagrangian of QED, $\mathcal{L}^{\text{QED}}_{\text{total}} = \mathcal{L}^{(0)} + \kappa g(x)\mathcal{L}^{(1)}$, even vanishes: $s\mathcal{L}^{\text{QED}}_{\text{total}} = 0$. Our method yields

$$\Big[Q_\psi, T^{\text{tree}}_{n\,S_0}\Big(\big(\int g\,\mathcal{L}^{(1)}\big)^{\otimes n}\Big)\Big] = 0\ , \tag{5.1}$$

where $Q_\psi \stackrel{\text{def}}{=} \int d^3x\,(\bar{\psi}\gamma^0\psi)_{S_0}$, cf. Appendix B of [7].

Can the method of this paper be applied to *massive* spin-2 gauge fields? This amounts to the question whether there is a BRS-invariant Lagrangian for such fields with the properties (3.1)-(3.5) and (2.17).

In [19] the requirement of PGI-tree has been applied to *massive* spin-2 fields. Starting with the most reasonable free theory (in Feynman gauge) and proceeding similarly to [28, 27] the lowest orders $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ of the possible interaction $\mathcal{L} = \sum_{n=1}^{\infty} \kappa^n \mathcal{L}^{(n)}$ have been derived. The result agrees with the corresponding terms of the Einstein-Hilbert Lagrangian with cosmological constant $\Lambda \neq 0$: $\mathcal{L}_{E\Lambda} := \mathcal{L}_E - \frac{2}{\kappa^2} \sqrt{-g}\, \Lambda$. The mass m of the free spin-2 field is related to Λ by $m^2 = 2\,\Lambda$.

$\mathcal{L}_{E\Lambda}$ satisfies the BRS-invariance (4.47), too. But the expansion of $\mathcal{L}_{E\Lambda}$ in powers of κ contains a term $\frac{-1}{\kappa}\,\Lambda\, h$, which cannot be removed by adding a divergence, and the field equation to order κ^{-1} is nonsense: $\Lambda = 0$. (This contradiction is due to the fact that the Minkowski metric $\eta^{\mu\nu}$ is not a solution of the field equation to $\mathcal{L}_{E\Lambda}$.) Therefore, it seems that the formalism of this paper cannot be applied to $\mathcal{L}_{E\Lambda}$; omission of the term $\frac{-1}{\kappa}\,\Lambda\, h$ destroys the BRS-invariance of $\mathcal{L}_{E\Lambda}$.

It is doubtful whether the mass can be generated via Higgs mechanism (as for spin-1 fields), since we are not aware of such a procedure in the literature and since in [19] it has turned out that PGI-tree can be satisfied up to second order without introducing a physical Higgs field.

Appendices

A Algebraic off-shell formalism

This Appendix is a short introduction to the formalism given in [9] and [10]. Let φ be a complex scalar field in 4 dimensions. The classical phase space is $\mathcal{C} \stackrel{\mathrm{def}}{=} \mathcal{C}^\infty(\mathbb{R}^4, \mathbb{C})$. The classical field $(\partial^a \varphi)(x)$, $a \in \mathbb{N}_0^4$, is the evaluation functional

$$(\partial^a \varphi)(x) \;:\; \mathcal{C} \longrightarrow \mathbb{C}\;,\; (\partial^a \varphi)(x)(h) = \partial^a h(x)\;, \quad \varphi^\star(x)(h) = \overline{h(x)}\;. \qquad \text{(A.1)}$$

Let $\mathcal{F}$ be the set of all functionals[8] $F \equiv F(\varphi) : \mathcal{C} \longrightarrow \mathbb{C}[[\hbar]]$ which have the form

$$F(\varphi) = \sum_{n=0}^{N} \int dx_1 \dots dx_n \,\varphi(x_1) \cdots \varphi(x_n) f_n(x_1, \dots, x_n)\;, \qquad N < \infty\;, \qquad \text{(A.2)}$$

[8] $\mathbb{C}[[\hbar]]$ is the space of all formal power series $\sum_{l=0}^{\infty} c_l \hbar^l$ with $c_l \in \mathbb{C}$.

where $f_0 \in \mathbb{C}[[\hbar]]$. The higher f_n's are $\mathbb{C}[[\hbar]]$-valued distributions with compact support, which are symmetric under permutations of $x_1, ..., x_n$ and which are 'translation invariant up to smooth functions', i.e.

$$WF(f_n) \subset \{(x,k) \mid \sum_{i=1}^{n} k_i = 0\} \; . \tag{A.3}$$

Due to (A.1) it holds $F(\varphi)(h) = F(h)$. $\mathcal{F}$ is a *-algebra with the classical product $(F_1 \cdot F_2)(h) := F_1(h) \cdot F_2(h)$ and $\langle f_n, \varphi^{\otimes n}\rangle^\star = \langle \overline{f_n}, (\varphi^\star)^{\otimes n}\rangle$. The functional $\omega_0 : \mathcal{F} \to \mathbb{C}[[\hbar]]$, $\omega_0(\sum_n \langle f_n, \varphi^{\otimes n}\rangle) = f_0$ is the 'vacuum state'.

The support of $F \in \mathcal{F}$ is the support of $\frac{\delta F}{\delta \varphi}$. Let $\mathcal{P}$ be the algebra of all polynomials in φ, φ^* and their partial derivatives. The vector space $\mathcal{F}_{\mathrm{loc}}$ of *local* functionals is the set of all $F \in \mathcal{F}$ of the form

$$F = \int dx \sum_{i=1}^{N} A_i(x) h_i(x) \; , \quad A_i \in \mathcal{P} \; , \; h_i \in \mathcal{D}(\mathbb{R}^4) \; . \tag{A.4}$$

Given an action S, the set $\mathcal{C}_S \subset \mathcal{C}$ is the set of all smooth solutions of the Euler-Lagrange equations belonging to S with compactly supported Cauchy data. We set $F_S \stackrel{\mathrm{def}}{=} F|_{\mathcal{C}_S}$, $F \in \mathcal{F}$. We study actions of the form $S_{\mathrm{total}} = S_0 + S_{\mathrm{int}}$ with free part S_0 and compactly supported interacting part: $S_{\mathrm{int}} \in \mathcal{F}_{\mathrm{loc}}$. We always assume that S_{total} is such that the Cauchy problem has a unique solution. Given two actions $S^{(j)}_{\mathrm{total}} = S_0 + S^{(j)}_{\mathrm{int}}$ $(j = 1, 2)$ of this kind, there exists a map

$$r_{S^{(1)}_{\mathrm{total}}, S^{(2)}_{\mathrm{total}}} \; : \; \mathcal{C}_{S^{(2)}_{\mathrm{total}}} \longrightarrow \mathcal{C}_{S^{(1)}_{\mathrm{total}}} \; , \; f_2 \mapsto f_1 \; , \tag{A.5}$$

such that f_1 agrees with f_2 outside the future of $(\mathrm{supp}\, \frac{\delta S^{(1)}_{\mathrm{int}}}{\delta \varphi} \cup \mathrm{supp}\, \frac{\delta S^{(2)}_{\mathrm{int}}}{\delta \varphi})$. A *retarded field*

$$A^{\mathrm{ret}}_{S_{\mathrm{int}}}(x) \stackrel{\mathrm{def}}{=} A(x) \circ r_{S_0 + S_{\mathrm{int}}, S_0} \; : \; \mathcal{C}_{S_0} \longrightarrow \mathbb{C} \; , \quad A \in \mathcal{P} \; , \tag{A.6}$$

is a functional on the free solutions which solves the field equation for $S_{\mathrm{total}} = S_0 + S_{\mathrm{int}}$ if $A = \varphi$ or φ^*. The perturbation expansion of a classical interacting field is the Taylor series of the corresponding retarded field as functional of S_{int},

$$A^{\mathrm{ret}}_{S_{\mathrm{int}}}(x) \simeq \sum_{n=0}^{\infty} \frac{1}{n!} R^{\mathrm{class}}_{n,1\, S_0}(S^{\otimes n}_{\mathrm{int}}, A(x)) \equiv: R^{\mathrm{class}}_{S_0}\Big(e_{\otimes}^{S_{\mathrm{int}}}, A(x)\Big) \; , \tag{A.7}$$

where $R^{\rm class}_{n,1\,S_0} : \mathcal{F}_{\rm loc}^{\otimes(n+1)} \to \mathcal{F}|_{\mathcal{C}_{S_0}}$ is the classical retarded product. (The lower index S_0 of $R^{\rm class}_{S_0}$ is redundant, however we write it in view of the conventions in QFT.)

Next we study the quantization of the free theory. Let $\Delta_+^{(m)}$ be the 2-point function of the free scalar field with mass m. We define a $*$-product $\star \equiv \star_m$ on $\mathcal{F}$ [1],

$$(F \star_m G)(\varphi) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \int dx_1 \dots dx_n dy_1 \dots dy_n \frac{\delta^n F}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)} \\ \cdot \prod_{i=1}^{n} \Delta_+^{(m)}(x_i - y_i) \frac{\delta^n G}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)} \ , \tag{A.8}$$

which is associative and non-commutative. Let $\mathcal{J}^{(m)} \subset \mathcal{F}$ be the ideal (with respect to the classical product) generated by the free field equation $(\square + m^2)\varphi = 0$. Let $\mathcal{F}_0^{(m)} \equiv \mathcal{F}/\mathcal{J}^{(m)}$. Due to $(\square + m^2)\Delta_+^{(m)} = 0$ the $*$-product (A.8) induces a well defined product $\mathcal{F}_0^{(m)} \times \mathcal{F}_0^{(m)} \to \mathcal{F}_0^{(m)}$, and we denote the corresponding algebra by $\mathcal{A}_0^{(m)} \equiv (\mathcal{F}_0^{(m)}, \star_m)$. The latter can be faithfully represented on Fock space, the $*$-product goes over into the operator product and, in addition, the classical product into the normally ordered product. ω_0 induces a state on $\mathcal{A}_0^{(m)}$ which corresponds to the Fock vacuum.

Finally we turn to perturbative QFT. Let $F, S_n \in \mathcal{F}_{\rm loc}$ with $F, S_n \sim \hbar^0$ ($n \in \mathbf{N}$), and let $S \equiv S(\kappa) = \sum_{n=1}^{\infty} \kappa^n S_n$ be a formal power series with $S(0) = 0$. We associate to (F, S) a formal power series

$$F_{S/\hbar} = \sum_{n=0}^{\infty} \frac{1}{n!} R_{n,1}((S/\hbar)^{\otimes n}, F) \equiv: R(e_\otimes^{S/\hbar}, F) \tag{A.9}$$

which we interpret as the functional F of the interacting retarded field under the influence of the interaction S where κ is the expansion parameter of the formal power series. The retarded products

$$R_{n,1} \ : \ \mathcal{F}_{\rm loc}^{\otimes(n+1)} \longrightarrow \mathcal{F} \tag{A.10}$$

are *linear* maps which are *symmetric in the first n factors.* Their basic properties are

- zeroth order $\quad R_{0,1}(F) = F$,

- Causality

$$\text{supp}\, R_{n,1} \subset \{(x_1,\ldots,x_n,x) \in \mathbb{R}^{4(n+1)} \mid x_i \in x + \overline{V}_-,\ \forall i = 1,\ldots n\} \tag{A.11}$$

- and the GLZ relation

$$i\Big[R\Big(e_\otimes^{S/\hbar},F\Big),R\Big(e_\otimes^{S/\hbar},H\Big)\Big]_\star = R\Big(e_\otimes^{S/\hbar}\otimes H,F\Big) - (H\leftrightarrow F)\ . \tag{A.12}$$

We set $R_{S_0}(...) \stackrel{\text{def}}{=} R(...)|_{\mathcal{C}_{S_0}}$.

$R^{\text{tree}}_{n,1}$ is the contribution of the tree diagrams to $R_{n,1}$. As shown in Sect. 5.2 of [8], $R^{\text{tree}}_{n,1}$ is that part of $R_{n,1}$ with the lowest power of $\hbar$, more precisely

$$R_{n,1}(F_1,...,F_{n+1}) = R^{\text{tree}}_{n,1}(F_1,...,F_{n+1}) + \mathcal{O}(\hbar^{n+1}) \quad \text{and} \quad R^{\text{tree}}_{n,1}(F_1,...,F_{n+1}) \sim \hbar^n \tag{A.13}$$

if $F_1,...,F_{n+1}\sim\hbar^0$.

B Zeroth order of the BRS-current for spin-2 gauge fields

From $\mathcal{L}^{(0)}$ (4.43) one obtains the free field equations

$$\Box\, h^{\mu\nu}_{S_0} = 0\ , \quad \Box\, u^\mu_{S_0} = 0\ , \quad \Box\, \tilde{u}_{\mu\, S_0} = 0\ . \tag{B.1}$$

By using (3.11) and (4.49) we obtain

$$j^{(0)\,\mu} = -\frac{\partial\mathcal{L}^{(0)}}{\partial\varphi_{i,\mu}}\, s_0\,\varphi_i - \partial_\nu\,\mathcal{D}^{(-1)\,\nu}\,u^\mu - s_0\,\mathcal{D}^{(0)\,\mu} - s_1\,\mathcal{D}^{(-1)\,\mu} + F^{(0)\,\mu}\ , \tag{B.2}$$

where we have taken into account $\mathcal{L}_{\rm E}^{(-1)} = \partial_\mu\, \mathcal{D}^{(-1)\,\mu}$. After restriction to $\mathcal{C}_{S_0}$ (B.1), we obtain the following non-vanishing contributions to $(-j_{S_0}^{(0)\,\mu})$:

$$
\begin{aligned}
\Big(\frac{\partial \mathcal{L}_{\rm E}'^{(0)}}{\partial h^{\alpha\beta}_{,\mu}}\, s_0\, h^{\alpha\beta}\Big)_{S_0} &= \frac{1}{2}\, h_{\alpha\beta\, S_0}^{,\mu}\, (u_{S_0}^{\alpha,\beta} + u_{S_0}^{\beta,\alpha}) - h_{S_0}^{\alpha\mu,\beta}\, (u_{\alpha,\beta\, S_0} + u_{\beta,\alpha\, S_0} - \eta_{\alpha\beta}\, u^\lambda_{,\lambda\, S_0})\ , \\
\Big(\frac{\partial \mathcal{L}_{\rm GF}}{\partial h^{\alpha\beta}_{,\mu}}\, s_0\, h^{\alpha\beta}\Big)_{S_0} &= h_{\alpha\gamma\, S_0}^{,\gamma}\, (u_{S_0}^{\alpha,\mu} + u_{S_0}^{\mu,\alpha} - \eta^{\alpha\mu}\, u^\lambda_{,\lambda\, S_0})\ , \\
\Big(\frac{\partial \mathcal{L}_{\rm ghost}^{(0)}}{\partial \tilde{u}_{\nu,\mu}}\, s_0\, \tilde{u}_\nu\Big)_{S_0} &= (u^\mu_{,\tau\, S_0} + u_{\tau\, S_0}^{,\mu})\, h^{\tau\rho}_{,\rho\, S_0} - u^\lambda_{,\lambda\, S_0}\, h^{\mu\rho}_{,\rho\, S_0}\ , \\
\big(\partial_\nu\, \mathcal{D}^{(-1)\,\nu}\big)_{S_0}\, u^\mu_{S_0} &= h^{\alpha\beta}_{,\alpha\beta\, S_0}\, u^\mu_{S_0} \\
\big(s_0\, \mathcal{D}^{(0)\,\mu} + s_1\, \mathcal{D}^{(-1)\,\mu}\big)_{S_0} &= \frac{1}{2}\, h_{,\nu\, S_0}\, (u_{S_0}^{\mu,\nu} + u_{S_0}^{\nu,\mu}) + \frac{1}{2}\, h_{S_0}\, u^{\lambda,\mu}_{,\lambda\, S_0} - \frac{1}{2}\, (h_{S_0}\, u^\lambda_{S_0})^{,\mu}_{,\lambda} \\
&\quad - h^{\mu\nu}_{S_0}\, u^\lambda_{,\lambda\nu\, S_0} + (h^{\rho\nu}_{S_0}\, u^\mu_{,\nu\, S_0})_{,\rho} - (h^{\mu\nu}_{,\nu\, S_0}\, u^\rho_{S_0})_{,\rho} \\
-\big(F^{(0)\,\mu}\big)_{S_0} &= -h^{\tau\nu}_{,\nu\, S_0}\, u^\mu_{,\tau\, S_0} - h^{\tau\nu}_{,\nu\, S_0}\, u_{\tau\, S_0}^{,\mu} + h^{\mu\nu}_{,\nu\, S_0}\, u^\lambda_{,\lambda\, S_0}\ . \qquad \text{(B.3)}
\end{aligned}
$$

The sum of these terms is equal to (4.50).

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